This book represents the culmination of a life in which two major passions have joined: the passion for music and the passion for mathematics.

Mihai Brediceanu is one of the major Romanian conductors of the second half of the twentieth century. During his brilliant 50-year international career, he conducted operatic and symphonic music in thirty-two countries and twenty capitals of the world, from Paris to Lima and from London to Beijing. He led important musical establishments such as the "George Enescu" Philharmonic in Bucharest, the Bucharest Opera, the Istanbul Opera, the Syracuse, NY, Symphony Orchestra and the "George Enescu" International Festival as either General or Musical Director.

For his activity, he was awarded the prestigious decorations "L'Ordre National des Arts et Lettres" by the Ministry of Culture of France and "Steaua României" by the President of Romania.

During this whole period he was concerned with establishing a mathematical order in the ineffable world of music. He delivered papers at several international mathematics symposiums. He also held lectures at reputable universities around the world, such as Harvard, Columbia, the Tokyo University and the Bucharest Conservatory.

In 1974, he was granted the Ph.D. degree in mathematics by the University of Bucharest. He invented and patented the "Polytimer", a revolutionary multi-channel metronome, which opens new horizons to composers by allowing the performance of complex music written in simultaneous but different tempos.

"The Topology of Sound Forms and Music", written in a concise yet comprehensive and accessible manner, is a book addressed both to mathematicians and to musicians who are offered a new approach on the intimate relationship between music and mathematics, as well as new methods of composition and new paths for exploring the musical language.

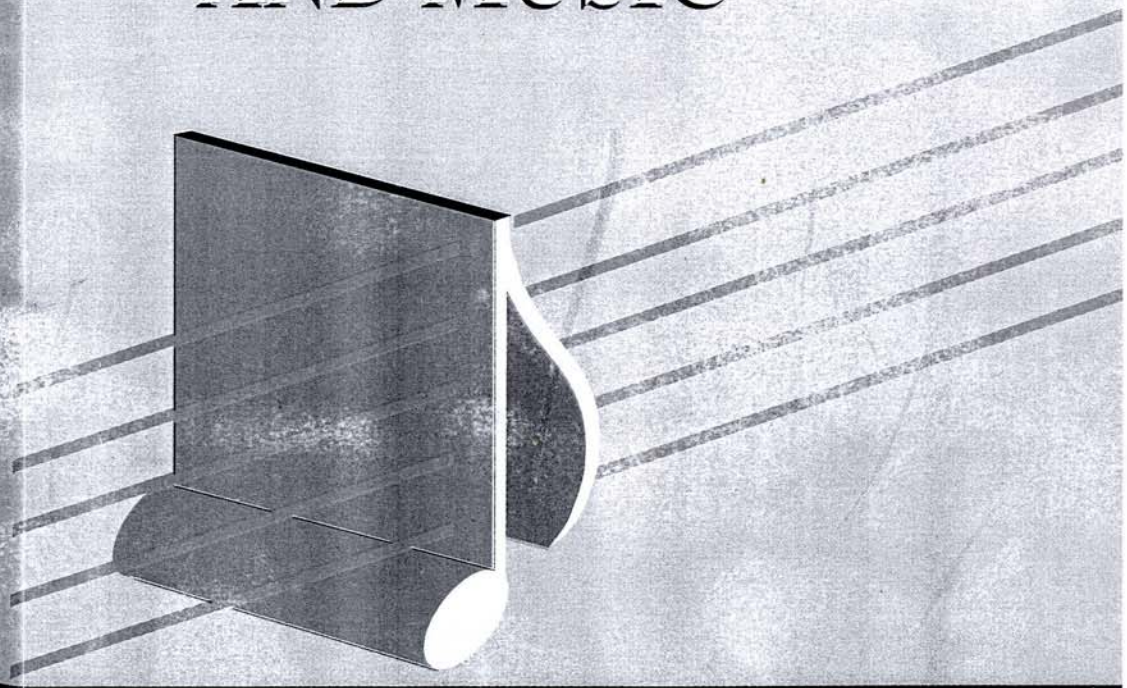
ISBN 973-27-0855-7

MIHAI BREDICEANU

TOPOLOGY OF SOUND FORMS AND MUSIC

MIHAI BREDICEANU

TOPOLOGY OF SOUND FORMS AND MUSIC



EDITURA ACADEMIEI ROMÂNE

THE BOOK WAS PUBLISHED WITH SUPPORT
FROM THE ROMANIAN MINISTRY OF CULTURE AND
CULTS

MIHAI BREDICEANU

TOPOLOGY OF SOUND FORMS AND MUSIC

EDITURA ACADEMIEI ROMÂNE
București, 2002

Copyright © 2002 EDITURA ACADEMIEI ROMÂNE

All rights reserved. This book is protected by copyright. No part of it may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photocopying, recording, scanning or otherwise, without the prior permission of the Publisher.

Editorial assistant: RODICA FLORESCU

Technical editor: DIANA RUSU

Graphics: VLAD VOICULESCU,
EMANUEL BORCESCU,
VIRGIL HRIȚCU,
DAN NEDELCU

Cover: GIGI GAVRILĂ

ISBN 973-27-0855-7

EDITURA ACADEMIEI ROMÂNE
(THE PUBLISHING HOUSE OF THE ROMANIAN ACADEMY)
Calea 13 Septembrie nr. 13, Sector 5
RO-76117, BUCHAREST, ROMANIA
Tel. 40-21-411.9008; Fax 40-21-410.3983
e-mail: edacad@ear.ro

CONTENTS

FOREWORD 15
INTRODUCTION 17

Part one. ELEMENTS OF A THEORY OF SOUND FORMS 25

Chapter 1. Sound events, forms and spaces 25

1.1.1. Phenomenological aspects of sound events and sound forms 25
1.1.2. Definition 27
1.1.3. Sound space. Formalization of the concepts of sound point, sound event and sound form. Null and imaginary sound events 27
1.1.4. Space representation of the sound form 30
1.1.5. Real sound space. Representation of sound forms in the real sound space 31
1.1.6. Sound form space. Dimensions of the sound event and of the sound form 32
1.1.7. Virtual sound space. Virtual sound forms 34

Chapter 2. Sound forms and topological transformations 34

1.2.1. Sound forms and the concept of topological transformation 34
1.2.2. Musical sounds: curves in the sound space $\mathbf{R}^2$ 36

Chapter 3. From the general sound event to spoken languages and musical structures 38

1.3.1. Classification of sound forms: signs and signals 38
1.3.2. The role of sound forms in natural spoken languages. Two modes of transmitting information in spoken languages: the logic-rational language and the emotional language. Code and resonance 39
1.3.3. Emotional modulation of the logic-rational spoken language 41
1.3.4. Combinatorial and parametric aspects in spoken languages 42
1.3.5. Combinatorial systems and topological transformations used for transmitting information through sound forms. Combinatorial systems of word generation 42
1.3.6. Topological space structure of sound forms in spoken languages 43
1.3.7. Topological transformations mapped to sound forms representing words and phrases. Artistic interpretation of a text considered as a topological transformation of the respective text sound form 44
1.3.8. Graphical aspect of natural languages. Writing: a filter of information conveyed through topological transformations applied to the sound forms of spoken languages 45

Chapter 4. The role of sound forms in the musical language 46

1.4.1. A higher degree of text emotional modulation: the song 46
1.4.2. Musical "modes": "containers" of finite sets of pitch values 47
1.4.3. A "container" holding a finite set of musical duration 47
1.4.4. Song sound space structure 47
1.4.5. The act of performing a song: topological transformation of the sound form representing the considered song 48

1.4.6. The musical instrument associated to human voice 48
1.4.7. A language exclusively working through "resonance": the musical instrumental language. Detachment and autonomy of the emotional modulation 49
1.4.8. Sets of timbres, intensities values and positions in the usual Euclidean space. The sound space with the deepest structure: the musical instrumental language space 49
1.4.9. The musical instrumental language: a Cartesian product of finite sets representing values of sound parameters 51
1.4.10. An attempt to form a hierarchy of sound languages depending on the space structure of the sound forms which represent these languages 51
1.4.11. A possible hierarchy of natural spoken languages in terms of the value of the ratio concerning the amount of information conveyed through the combinatorial aspect and the amount of information conveyed through the parametric aspect of the word 52
1.4.12. Two types of topological transformations of sound forms in the European musical language 53

Chapter 5. The role of semiography in the development of European music 54
1.5.1. Creative aspect of the written sign: graphical thinking 54
1.5.2. The double function of musical semiography: means of notation and creation tool.. 55
1.5.3. Abstraction of the act of musical creation: the composer works with the image of the sound 56
1.5.4. The staff: ancestor of Cartesian coordinates. Centuries ago, composers had the intuition of multi-dimensional spaces 56
1.5.5. Two ancestors of the mathematical functions: "counterpoint" and "ciphered bass". 58
1.5.6. The composer and the staff geometry. The concept of "transformation" appeared in music as early as the 13th century 60
1.5.7. Combinatorial aspects in structuring the musical time 61
1.5.8. Chronological time value of the symbols denoting duration 63
1.5.9. Conclusions 63

Part two. MUSICAL CONCEPTS AND COMPOSITION METHODS INTERPRETED IN REAL SOUND SPACE R^2 AND IN VIRTUAL SOUND SPACE R^3 65

Chapter 1. Musical sounds and modes in sound space R^2 65
2.1.1. Musical sounds represented in sound space $\mathbf{R^2}$ 65
2.1.2. Rectilinear sounds considered as vectors in sound space $\mathbf{R^2}$. "Glissando" sounds. Clusters 66
2.1.3. Representation of musical sound forms by matrices 70
2.1.4. The value of a pitch interval may be expressed by the absolute value of the difference between two sound point ordinates 71

Chapter 2. Topology of modes in real sound space R^2 73
2.2.1. Continuous structure of sound space $\mathbf{R^2}$. Discontinuous structure of musical sound space. A short journey into the world of modes 73

2.2.2. Different units of measurement for intervals separating the modes' consecutives steps 73
2.2.3. The purpose of the journey in the world of modes revealed. Musical modes viewed as topologies in the sound space $\mathbf{R^2}$ 74
2.2.4. The set of modes with a number *n* of sounds is of the power of continuum. Related to the respective number the set forms a class of equivalence 76
2.2.5. Mode representation through vectors 77
2.2.6. Difficulty of including in discrete spaces and by combinatorial methods the various modal categories in a general unitary system 78
2.2.7. Towards a possible general theory of musical modes by the instrument of continuous space topology 79
2.2.8. A set of still unexplored modes may become "building material" for the composer 79

Chapter 3. Musical transformations in sound space R^2 endowed with topological, vectorial and projective structures 80
2.3.1. Mapping of a sound of a certain image into another sound of a different image, amounts to a topological transformation of a curve in $\mathbf{R^2}$ 80
2.3.2. Transformation of a sound in musical notation and the corresponding operation in sound space $\mathbf{R^2}$. Definition vector. Transformation vector 83
2.3.3. Augmentation and diminution of a musical pitch interval represent topological transformations in sound space $\mathbf{R^2}$ 85
2.3.4. Polyphony considered as families of topological transformation of a curve representing a "cantus firmus" (1.5.5) 86
2.3.5. The mathematician-composer of the 11th century 90
2.3.6. Heterophony considered as families of topological transformations of a curve. Transformations bound to observe certain specific restrictions 90
2.3.7. Transformation of musical sound forms within the composition method. So-called "variations on a certain theme", interpreted as topological transformations. 91
2.3.8. The "modulation", a particular case of topological transformation used in music 93
2.3.9. Modal adjustement (MA): transformation of a sound form defined in $\mathbf{R^2}$ continuous sound space, into a sound form defined in a space with a certain modal structure 94

Chapter 4. Transformations in sound space R^2 endowed with affine structure 96
2.4.1. Transformation by translation 96
2.4.2. Transformations through homothety 104
2.4.3. A musical chord may be expressed in terms of a center and a family of homothety ratios 107
2.4.4. Reversion of chords expressed by homothety transformations 107
2.4.5. Transformations through parallel homology 108
2.4.6. "Ciphered bass" expressed by ratios of parallel and orthogonal homology 111
2.4.7. Reversion, inversion, reversion of inversion and identical transformation of a sound form constitute a Klein commutative group structure (1.5.7). This group can be interpreted as an affine transformation subgroup in sound space $\mathbf{R^2}$ 113

2.4.8. Group T interpreted as a subgroup of the affine group of transformations. A musical theme reversion and inversion interpreted as transformations by parallel homology. Reversion of inversion interpreted as a transformation by homothety 115

Chapter 5. Transformations in sound space R^2 endowed with Euclidean structure 117

2.5.1. Transformation through rotation 117

Chapter 6. Transformations in virtual sound space R^3 120

2.6.1. Transformations through translation composed with projection 120

2.6.2. Numerical example 122

2.6.3. The delay of the appearance moment of a sound S (case $m < 0$, $n = 0$) interpreted as its projection from $\mathbf{R}^3$ into $\mathbf{R}^2$ from a projection center Cp of coordinates $x(Cp) = x(D)$; $y(Cp) = y(A,D)$; $z(Cp) = u \quad u < 0$ 123

2.6.4. Modification of the disappearance moment of a sound expressed as function of parameters m, n, h and $l(u)$ 123

2.6.5. Sound pitch modification interpreted as a transformation by projection in the virtual sound space $\mathbf{R}^3$ 125

2.6.6. The value of a pitch interval (I) may be expressed as $I = z(S_v) * \text{tg}\alpha$ where $z(S_v)$ is the "height" to which a sound S has been "raised" in the virtual sound space $\mathbf{R}^3$, and α represents the angle of the orthogonal cylindrical projections with the oblique cylindrical projections when they intersect S_v 127

2.6.7. Simultaneous transformations of duration and pitch of a sound 128

2.6.8. A musical chord may be expressed by a family of projection angles 129

2.6.9. Transformations by translation composed with rotation and projection 129

2.6.10. Group T (2.4.8) interpreted as a subgroup of the group of Euclidean space transformations 130

Part three. BEHAVIOUR OF SOUND FORMS UNDER THE GENERALIZED ACTION OF TOPOLOGICAL TRANSFORMATIONS WHICH EXHIBIT PARTICULAR CASES IN MUSICAL COMPOSITION METHODS. A POSSIBLE EXTENSION OF THESE METHODS 135

Chapter 1. The composer and the topological transfomations 135

3.1.1. Some conclusions regarding the preceding chapters 135

3.1.2. Ways of notating the parameters of some transformations and special signs for defining particular musical sounds 136

3.1.3. Compound transformations useful to the composer in the process of structuring the musical discourse. Ordinates lining up (OL) and abscissae ordering (AO) 138

3.1.4. Using modal adjustment 141

Chapter 2. Behaviour of sound forms under the action of homology transformation 142

3.2.1. Homology with its center at finite distance 142

3.2.2. Case I (sound form situated between center and axis) 143

3.2.3. Case II (axis situated between center and sound form) 150

3.2.4. Case III (center between axis and form) 156

3.2.5. Parallel homology (center situated at infinity) 163

3.2.6. Transformations through parallel homology of $q = -1$ value, which in certain situations of the axis and projections (A.3.8), cause the reversion and inversion used in music (2.4.4, 2.4.5) 171

3.2.7. A story with J.S.Bach, fugues "in mirror" and homologies 176

Chapter 3. Behaviour of sound forms under the action of transformations through homothety 187

3.3.1. A part of the main theme of George Enescu's First Symphony subjected to homothety 187

Chapter 4. Behaviour of sound forms under the action of transformations through rotation 196

3.4.1. 196

3.4.2. Study of the behaviour of sound form F, defined by the matrix subjected to transformation through rotation 196

Chapter 5. Behaviour of sound forms under the action of transformations through inversion 211

Chapter 6. Transformations in sound space R^3. Behaviour of sound forms under the action of translation, rotation and projection. Compound transformations 214

3.6.1. General remarks 214

3.6.2. Taper projection 216

3.6.3. Case 1 ($zCp > zFp$) 216

3.6.4. Case 2 ($0 < zCp < zFv$) (Cp between Fv and $\mathbf{R}^2$) 222

3.6.5. Case 3 ($zCp < 0$)(Cp situated "below" sound space $\mathbf{R}^2$) 225

3.6.6. Case 4. Cylindrical projection ($zCp \to \infty$) 229

Part four. CONTINUOUS TOPOLOGICAL TRANSFORMATIONS AND GENERATIVE MECHANISMS 233

Chapter 1. Dynamic aspects in studying and generating sound forms 233

4.1.1. General remarks 233

4.1.2. Aristotle's vision of the correspondence between motion and form and its possible interpretation in the field of sound forms 234

4.1.3. Dynamic aspect of sound forms 234

4.1.4. Graphical representation of the dynamic aspect of sound forms. Sound form interpretation as generated by the track of a hyperplane in motion 235

4.1.5. The formative aspect of the musical language 235

4.1.6. Any sound event may be generated by the track of a sound point in motion in the real sound space 236

Chapter 2. Generating mechanism 237

4.2.1. Parameter values expressed in terms of time 237

4.2.2. Elementary and complex musical sound forms 237

4.2.3. Three modes of generating elementary music forms through a sound point moving in sound space $\mathbf{R}^2$ 238

4.2.4. The transformed segments of a sound trajectory produce a music fragment 245
4.2.5. A sound point through a compound transformation (translation compound with parallel homology) generates the accompaniment of a recitative belonging to a Bach cantata 246
4.2.6. A sound point in motion generates through the same composed transformations (translation, parallel homology) the *soprano* voice of Verdi's *Requiem* and the associated voices *alto, tenor, bass* 249
4.2.7. Musical fragments consisting exclusively of the images obtained through transformations applied to some points or segments belonging to a sound trajectory 251
4.2.8. The concept of "function" in traditional music 253
4.2.9. Transformation of a sound trajectory segment parallel to the time axis in an arpeggio with a certain tonal function 254
4.2.10. Generation of musical themes and their accompaniment through topological transformations. Can Mozart's name be involved in such methods? 255
4.2.11. Whereto leads the non-fulfilment of some conditions 256
4.2.12. A surprise for an enemy of mathematical thinking in music 259

Chapter 3. Musical structures generated by the discontinuous survey of a sound form in motion in R^2 260

4.3.1. Sound forms in motion 260
4.3.2. Mathematical analysis in sound space. The musician, the derivatives. Rolle's, Fermat's and Lagrange's theorems 261
4.3.3. Musical significance of the variation of function $(f(x) = 3(x^2 - 4x + 3))$ 263
4.3.4. The speed at a given moment of the unfolding of a glissando sound. The derivative assisting again the composer 264
4.3.5. A sound form "walking" on a curve in space $\mathbf{R^2}$ generates imitations and canons 265

Chapter 4. The concept of "function of time" as reflected in the thinking of preclassical composers. Sixteenth century music makes use of space coordinates 269

4.4.1. Open and closed canons 269
4.4.2. The composer confronted with interpolation problems 273
4.4.3. A polynomial of 4th degree "loaded" with Bernabei's "theme" can generate a large family of canons, that of the Italian composer's being a member of this family 273

Chapter 5. Generative mechanisms making use of continuous composed topological transformations. Parameters values depending on time *t*. New working tools for the contemporary composer 276

4.5.1. Continuous and discontinuous transformations of a musical sound form 276
4.5.2. Compound transformations. Parameters taking, depending on time *t*, continuous values in a given values interval 278
4.5.3. Continuous transfomation through translation composed with homothety. Homothety ratio value expressed in terms of musical time 278
4.5.4. A sound form continuously transformed through translation compound with homothety and rotation. A rotation angle expressed in terms of musical time ... 281

4.5.5. "Sound surfaces" made up of homeomorphic images obtained from an initial sound form 284
4.5.6. A generating mechanism producing a class of pieces, where a fragment of a piece by Anton Webern may also be found 284

Chapter 6. Generative mechanisms in virtual sound space R^3 297

4.6.1. A generative mechanism which, along with the practically infinite set of polyhomeorphic structures it generates, also creates the basic structure of the *Two-Voice Invention* in C Major by J.S. Bach 297
4.6.2. Bach's "motif" subjected to the action of generative mechanism 301

Chapter 7. Two aspects of time in generative mechanisms: chronological time, musical time 311

4.7.1. A rotation angle expressed in terms of chronological time 311
4.7.2. A space with two time reference systems: musical time and chronological time 313
4.7.3. The homothety or homology centres and ratios may also be expressed in terms of the chronological time 316
4.7.4. Dual time (chronological and musical) generative mechanisms in space R^3 318
4.7.5. In order to carry out the small piece "Fantasia on a sinusoid", the composer should remember the integrals 323

Part five. ROLE OF COMPUTERS IN GENERATING MUSICAL SOUND STRUCTURES BY MEANS OF TOPOLOGICAL TRANSFORMATIONS AND OF GENERATIVE MECHANISMS 328

Chapter 1. General remarks 328

5.1.1. The important amount of calculations needed when dealing with topological transformations applied to sound forms, requires the help of the computer 328
5.1.2. Basic operations in connection with computer utilization in sound form transformations 328

Chapter 2. Brief description of some programs 330

5.2.1. HOMOLOGY 330
5.2.2. HOMOPAL 330
5.2.3. HOMOTHETY 331
5.2.4. ROTATION 331
5.2.5. SPACE 331
5.2.6. SMOOTH 332
5.2.7. COPITCH 332
5.2.8. MODADJUST 332
5.2.9. SHIFT 333
5.2.10. SPREAD 334

Chapter 3. Programs operating with compound transformation 334

5.3.1. TRAJDATA and TRAJECT 334
5.3.2. MOTION 338

Chapter 4. Program translating in score symbols musical sound forms represented numerically in sound space R^2 339

5.4.1. PART 339

Sixth part. MUSIC AND POLYMODULAR TIME (STRUCTURAL POLYTEMPI MUSIC) 341

Chapter 1. General remarks and some considerations regarding musical time 341

Chapter 2. Polymodular time 342

6.2.1. Basic ideas concerning the polymodular time 342
6.2.2. Introductory concepts 343
6.2.3. Definitions 344
6.2.4. Degree of complexity 346
6.2.5. Some basic degrees of coherence of polymodular time 348
6.2.6. Spontaneous fundamental pulsations systems 352
6.2.7. Degrees of coherence at the level of the time reference systems in $[PMT]^n$, $n > 1$. 352
6.2.8. Examples of polymodular time structures with degree of complexity n, $(n > 1)$ 352

Chapter 3. Laws of distribution of virtual and real pulsations in a polymodular time structure (PMT) 354

6.3.1. Case of structures with degree of complexity 1 354
6.3.2. Tempo transformation 356
6.3.3. Case of structures with degree of complexity n, $n > 1$ 358

Chapter 4. Polymodular time and chronological time 358

6.4.1. Chronological time viewed as a time reference system $[TRS]^{n+1}$ for a polymodular structure $[PMT]^n$ 358

Chapter 5. Topological aspects of the polymodular time 360

Chapter 6. The polymodular time structure of music 361

6.6.1. Traditional music 361
6.6.2. Polytempi music 365
6.6.3. Music with polymodular time structures of degree of complexity 2 366

Chapter 7. Polytimer 373

ADENDDA

A.1. Introductory concepts of topology 376

A.1.1. Topological spaces, mappings and invariants 376
A.1.2. Compound transformations 379
A.1.3. Topological transformation of functions 379

A.2. Intuitive introduction of some fundamental topological concepts 381

A.2.1. Neighbourhood, continuity, connectivity, bicontinuous and one-to-one transformations 381
A.2.2. Topological transformations and rubber membrane geometry 383

A.3. Various structures of topological spaces: projective, affine, Euclidean and metric 385

A.3.1. Fundamental remarks 385

A.3.2. Transformations in the general topological space 386
A.3.3. Transformations in topological space having a projective structure 387
A.3.4. Central projection 387
A.3.5. Parallel projection 388
A.3.6. Homology 389
A.3.7. Transformations in topological space having affine structure 390
A.3.8. Parallel homology 391
A.3.9. Translation 392
A.3.10. Homothety 392
A.3.11. Transformations in topological space having euclidean structure 393
A.3.12. Rotation 393
A.3.13. Isogonal transformations. Inversion 394
A.3.14. Transformations in topological space having a metric structure 395
A.3.15. Transformations in the topological space $\mathbf{R}^3$ endowed with projective structure 395
A.3.16. Homology in $\mathbf{R}^3$ 398
A.3.17. Transformations in the topological space $\mathbf{R}^3$ having affine structure 399
A.3.18. Transformations in the topological space $\mathbf{R}^3$ having Euclidean structure. Rotation in $\mathbf{R}^3$ 399
A.3.19. Rotation round an axis included in the same plane with the figure and intersecting it 400
A.3.20. Compound transformations in the topological space $\mathbf{R}^3$ 400

A.4. Analytical aspects of some foregoing considered transformations . 402

A.4.1. Projective transformations in $\mathbf{R}^2$ 402
A.4.2. Homology 403
A.4.3. Three relative positions of the homology axis, the homology center and the point to be transformed 404
A.4. 4. Relative position of the transformed image with respect to the value of the homology biratio 405
A.4.5. Graphical representation of the critical values of the homology biratio 406
A.4.6. Maintaining the continuity of a straight line segment 408
A.4.7. Particular case of a line segment transformation 412
A.4.8. Behaviour of a vector under the action of homology mapping 412
A.4.9. Study of the three cases of relative position with respect to homology axis and center 413
A.4.10. Behaviour of a geometric figure under the action of transformation through homology 415
A.4.11. Reversion and inversion of the transformed image 417
A.4.12. Affine transformations 418
A.4.13. Parallel homology 418
A.4.14. Translation 419
A.4.15. Homothety 420
A.4.16. Euclidean transformations 422
A.4.17. Inversion 422
A.4.18. Metric transformations 424

A.4.19. Projection from R^3 in R^2 424
A.4.20. Behaviour of a geometrical plane figure situated in the space R^3 under the action of transformation through projection from R^3 in R^2 425
A.4.21. Rotation in R^3 428

FOREWORD

I decided to write this book on the connections between sound forms and topology, being convinced that the organization of sounds in traditional musical structures represents just particular cases of much more complex systems.

I am sure that conceiving and achieving both a theory and the tools supporting this idea will actually make available infinite solutions to composers in their creative work. It is true that all this would have not been possible without modern technology. From the first ideas to their final form I had to travel a long path littered with a multitude of unknown factors and obstacles that I was able to overcome thanks to the help of a number of remarkable people.

The wrapping up of my ideas in a clear mathematical form was possible thanks to the support of Professor Wilbur LePage of Syracuse University and of Professor Miron Niculescu, member of the Romanian Academy, who was also my scientific advisor for the doctorate paper delivered in 1974.

My friend, Professor Solomon Marcus, made me familiar with the mathematical semiotics and its applications.

The computer programs, as well as the preparation of the examples in the book were the work of my collaborator Mrs. Otilia Marinescu.

The polytimer was successively developed in three versions by several teams of excellent technicians led by Professor Edward Stabler of Syracuse University and Mr. Ion Predonescu of ITC, Bucharest.

The collating, organization, record-keeping of diverse materials were carried out with dedication during all phases of the editing of this book by Mrs. Lucia Pappia.

My family stood close to me, encouraging and helping me in this daring enterprise.

I hereby wish to express my deepest gratitude to all those who helped and supported me in this effort.

THE AUTHOR

INTRODUCTION

The aim of this work has two distinct aspects:

1. the creation of an instrument of investigation allowing an adequate generalization, so that a number of seemingly dissimilar sound organizations may be included in a system, music being considered a sound organization belonging to that system;
2. the endowment of the musical composition act with new tools, by proving that the traditional methods are but particular cases of a much more general group of principles for sound organizations, out of which only a small part is used by composers.

The general frame of the processes to be considered in this work is given by the topological spaces and the topological transformations of these spaces.

The origin of the term 'topology' should be looked for in the Greek words 'topos' and 'logos' – place and speech, respectively. Therefore, topology appears as the branch of mathematics dealing with the place, the positions of the elements of a set within a space (not necessarily Euclidean) and the relations that can be established among these elements.

The act of modifying the place, hence the relations among the elements of the set, according to certain laws, while preserving some specific properties connected with the nature of these relations, leads to the concept of topological transformations.

The set elements are referred to as "points" and may range from members of a family to bridges connecting river isles or points of a geometric space. In the present work, the elements of the sets we are dealing with are the sound points, defined as irreducible elements forming sound events, particularly the musical sounds.

A rigorous mathematical presentation of some fundamental topological concepts, such as topological spaces, continuity, connectedness, neighborhood, invariant and topological transformations, is given in the *Addenda*. The reader less familiar with mathematical terms will be introduced to some of these concepts in a simpler, intuitive way. Of course, we are aware that the

intuitive understanding of some concepts cannot allow the grasping of their essence, nor access to the level of logical operations that would imply such concepts.

Harmony and polyphony treatises, as well as the works formalizing in a mathematical language their principles, deal with the process of composing *i.e.*, the organization of musical sounds, by making use of discrete combinatorial systems.

Unlike such approaches, the present work tries to grasp the overall, qualitative aspects proper to the sound structures participating in the transmission and transformation of the information by representing them in a continuous topological space. Music is examined from this point of view.

In order to attain these aims, the author works with the concept of *sound form* as a consciousness phenomenon, its formalization by means of a mathematical model; he also studies the transformations these forms are subjected to in the process of transmitting the emotional information carried by music.

By *sound form* we understand the set of relations established by human consciousness among a finite number of sound events. In its turn, the sound event is considered as the reflection in consciousness of an auditorily perceived wave phenomenon. Special interest is dedicated to the topological analysis of sound forms having musical significance and their behavior under the action of topological transformations.

The mathematical model of sound forms consists of an n-dimensional topological space, whose dimensions are the sound parameters and a family of connected sets of "sound points" belonging to this space. In this representation, the sound events show up as parts of Cartesian products of finite intervals defined on the coordinate axes, imaginable as parts of n-dimensional parallelepipeds. The Cartesian product of these parts of a finite set of such parallelepipeds leads to a connected set of sound points imaginable as part of an n-dimensional parallelepiped. This set represents the *sound form*, a subjective phenomenon resulting from the operations made by human consciousness when relationships are established among the values of the parameters of a number of sound events.

In order to simplify the research and explain the results, as well as to observe the tradition of musical theory treatises, the sound space is limited to a topological space $\mathbf{R}^2$ whose dimensions are time and pitch. This space is called the *real sound space.*

Also, for a better representation of transformational operations, it has been considered more advantageous to work with the sets of sound points themselves instead of carrying the concept of Cartesian product.

The continuity in time of sound forms, as well as the continuous character of their perception by human consciousness, have suggested the study of the effect generated by topological transformations applied to sound forms.

This study led to the idea of investigating the role of sound forms in the information transmission function in spoken natural languages considered as particular cases of sound forms.

As a result, it appears that the transmission of logic-rational information is obtained by combinatorial systems applied to some finite sets of sounds specific to a given natural language, while transmission of emotional information is obtained by topological transformations applied to sound forms initially obtained through combinatorial systems. These considerations allow a possible hierarchical classification of spoken languages, their positions in this hierarchy being determined by the ratio between the quantity of rational and emotional information carried by these sound forms. From this point of view, a filiation between spoken languages and music can be made.

Readers will then be able to follow how psycho-emotional information is transmitted and transformed and to discover the way to the most pure and complex psycho-emotional language, *music*. They will also be able to understand the sound language hierarchy induced by the profoundness of the sound space structure and the reason why music holds the highest degree in this hierarchy.

Reaching this point, they will be able to integrate easily different sound organizations into a general system, wherein music is a particular case only, even though it shines so vigorously over all the others.

The study of European music formation and development in terms of sound form lets us establish that, besides topological transformations occurring in the act of performing a musical discourse, a more elaborate category of such transformations may by detected in the act of musical composition. It is easy to prove that music, like any other branch of science, is to a great extent tributary of space representations and the use of graphic symbols. The musical stave and the symbols representing sounds, initially conceived as notation means, have very soon proved to be excellent creation tools.

The limited geometry of the musical stave, considered as a primitive system of coordinates of time and pitch values, has allowed the composer to raise musical composition to an abstract level and to think music by means of the *image of the sounds on paper*. This marked the beginning of the evolution of a great transformational system, materialized in rules of polyphony, harmony and composition of musical forms.

Instead of examining these musical aspects as isolated structure phenomena, obtained by combinatorial methods applied to some sets of musical sounds – the chromatic scale sounds, usually – we started the study of sound forms taking a *continuous space* as our field of operation where the respective structures occur as particular cases. In order to get a wide range of transformations that may be algebraically expressed, we associated to the real sound space $\mathbf{R}^2$ an operational space $\mathbf{R}^3$ referred to as *virtual sound space*. The sets of coplanar sound points located in $\mathbf{R}^3$ are considered *virtual sound forms*. The virtual sound forms are transformed within the virtual sound space $\mathbf{R}^3$ and then projected in the real sound space $\mathbf{R}^2$.

The methodology of this research consists in the study of the behaviour of sound forms when acted upon by different groups of transformations applied in sound spaces successively endowed with vectorial, projective, affine and Euclidean structures. This study demonstrates that a great number of known musical composition methods, among which the fundamental principles of polyphony, harmony and heterophony, as well as certain specific structuring systems in the counterpoint domain, may be considered as particular cases of topological transformations applied to sound forms consisting of musical sounds. This leads to the hypothesis that a practically infinite number of transformations of this type may be used for structuring the sound universe of contemporary music.

Readers, and composers especially, will easily discover that the traditional composition methods are largely enclosed in the topological transformation group and that this finding has as corollary a possible extension of these means in the process of musical creation.

While studying the behaviour of musical structures when acted upon by topological transformations, readers will meet mathematical "beings" with musical significance in the music of preclassic to contemporary composers. They are then reassured to step on known ground, wherefrom paths are detached, leading to still unknown places, sometimes full of magic.

The mathematical beings in question no longer live on a stave, but in topological spaces and have different names from those with the same faces in the musical world. For instance: modes and keys are particular topologies in sound spaces; the act of replacing a musical sound with another one of different pitch and duration, amounts to a topological transformation in the vectorial sound space $\mathbf{R}^2$; in the same space, pitch intervals, musical chords, as well as their transformations may be expressed by homothety ratios; transposition by a translation, reversion, inversion, diminution and augmentation of some musical themes are certain types of parallel and orthogonal homologies; an "inversion of reversion" is a rotation; the value

of a musical interval may be expressed by the value of the angle of a projection line with a plane in a tridimensional, virtual sound space...

But this is not only a matter of changing a name! The new name shows that the respective being belongs to a family with a great many members. A mathematician, through topological language, would say that the family is an equivalence class, a set of power of continuum. He will disclose to the composer that certain traditional composition tools have practically an infinite number of "faces" available to compose musical structures.

Some of the research carried out by the author referred to the "machine"-like aspect of musical creation and its particular mathematical logic, which, according to modern theses can be found in all types of human creative activities. For example, compound transformations applied to an initial sound form can be examined from a *kinetic* point of view, as the form undergoes continuous transformations in time. The homeomorphic images, which result from surveying at various moments the continuously transformed initial form, can then be linked to generate a musical piece in the mode of Bach's "inventions" but of course in a contemporary style.

Certain types of such generative techniques are studied, some of them generating, besides a practically infinite number of sound structures, certain musical forms as chorales, canons, inventions or fragments existing in Bach's, Bernabei's, Mozart's, or Anton Webern's music.

The number of possible generative techniques of significant musical interest is obviously immense. The possibility of associating to these techniques a generative theory is, also expected to appear. A composer's creative act consists then in selecting and utilizing the most appropriate of these generative methods.

The large number of calculations required when applying this theory, leads us to resort to computer assistance. A series of computer programs for generating topological transformations of sound forms have been devised.

The final part of this work deals with the concept of *polymodular time*. The author's Polymodular Time theory refers to the mathematical modelling of some time structures characterized by the property that time is simultaneously measured with various time intervals, referred to as "time modules", whose values are different and variable. This concept permits surveying, in parallel, a finite number of concomitant periodical and non-periodical unfolding events, inducing by their repeated occurrence different time units of measure. It also allows the study of some complex time structures, existing in nature, as physical, biological, astronomical, social or technological phenomena.

In music, the polymodular time concept permits solving the problem of composing polytempi music *i.e.*, music endowed with "voices" or groups of voices simultaneously unfolding at different "tempi", according to different units of measure of musical time. As late as the end of the 19th century, this problem was a concern for composers such as Gustav Mahler and Charles Yves. Subsequently, many contemporary composers were also interested in it. As a rule, the attempts to superpose "layers" of music simultaneously unfolding at different tempi, were not directed towards obtaining rigorous sound structures; the results tend to more or less random organizations.

When well defined structures were aimed at, the composers involved did not succeed in going beyond some particular cases (as, for instance, correlation of some tempi whose metronomic values admit a common divisor, thus obtaining a periodical common "beat" for the different layers), all these cases generally implying a mutual dependency of the different considered tempi.

Even under these circumstances, accurate performance in concerts of such scores was not possible, since no adequate technology existed and the movements of several conductors could not be satisfactorily coordinated. The need to solve this obstacle prompted the author to devise the *Polytimer*, a computer intended to coordinate polytempi concerts and performances, whose prototype was developed at the Syracuse University in 1974. An industrial model of this device was manufactured at the "ITC" computer company, Bucharest.

In drawing up the concept of *polymodular time*, the following ideas and concepts were used:

The moment of occurrence of a phenomenon having a periodical or non-periodical repetitive character represents a "pulsation". This pulsation is referred to as a *real pulsation*. With sound structures, the real pulsation is defined as being the occurrence moment of a null or non-null sound event. Pulsations are represented by points within a topological space. A finite set of such pulsations strictly ordered in time represents a *pulsation system.*

A collection of real pulsation systems provided with a system of *virtual pulsations* represents a *structure of polymodular time*. The virtual pulsation system is a reference system for the collection of real pulsation systems. Depending on the kind of relations to be established between the different real and virtual pulsations, a polymodular time structure may have various degrees of coherence. The hierarchical organization of some collections of reference systems leads to polymodular time structures of various degrees of complexity. The music conceived in polymodular time structures is called *structural polytempi music.*

The first performances with structural polytempi music took place in Syracuse, N.Y. (1976), followed by those in Bucharest (1977) and Paris at the Centre Culturel Georges Pompidou (1978).

Before closing these introductory ideas, we would like to add a few words addressed to readers inclined to think that there is no link between music and mathematics. Of course, if we consider music an ineffable emotional language, its study belongs to the fields of phenomenology and aesthetics; but if we consider music as an artistic object which carries emotion, the study of its structure (*i.e.* of the organization of the sounds forming it) belongs to mathematics.

It is easy to prove that composers have always made use of mathematical concepts, though sometimes expressed in musical terms; it is surprising to discover that a lot of concepts considered to be far from music were present in the art of sounds before turning up in mathematics.

An example is the musical stave itself, which in fact is a certain type of a two-dimensional space whose dimensions are time and pitch. It appeared in music as far back as the 11th century, being introduced to mathematics in an explicit and generalized form as late as the middle of the 17th century by Descartes.

From the very beginning of polyphony, the mathematical concept of "point" has been present in music in the word counter*point* (the *contrapunctus* from Bach's *Art of Fugue)*, initially defined as "punctus contra punctum". For a composer the "point" is a sound represented by its image on the stave, and the "counterpoint science" is the science or art of associating a point to another point according to a law, a concept equivalent to the *function* in mathematical analysis.

We have defined counterpoint as a method of composition; but by counterpoint composers also mean the very image obtained by this method from a sequence of initial point-sounds referred to as "cantus firmus". In this case, it is obvious for mathematicians that they are dealing with a topological transformation and that the counterpoint-method is nothing but a homomorphism.

Also, when a composer "modulates" or "passes" from one key to another, he makes use of the "sounds common to both keys", hence, from the mathematical point of view, he deals with the "intersection" of the sound sets representing the respective keys; many other examples could follow.

Since harmony and polyphony treatises actually reflect the science of generating musical structures by defining the relations among sounds, more precisely, the *place* they hold in musical discourse, then it becomes clear that *Analysis Situs,* "the analysis of the place", as topology was initially

called by Riemann, is the most adequate modern instrument for studying the existing traditional musical structures, but also for finding for this art new creation tools.

As Pythagoras was the first to define the value of musical pitch intervals by the strings' length ratio, it is clear that the relationships of mathematical and musical thinking are of old standing. We will agree with the Middle Ages Universities, where music was not taught in "trivium", like grammar, rhetoric and dialectics, but in "quadrivium", together with arithmetics, geometry and astronomy, hence with the exact sciences. Examples are the chairs for mathematics and music established at Oxford and Paris Universities about 1230.

The tradition to reach the study of music by routes that pass across the field of mathematics was lost for several centuries, but it has been resumed these last decades. We have tried to follow one of these routes in this work. Therefore, let us begin the journey...

1. ELEMENTS OF A THEORY OF SOUND FORMS

1.1. SOUND EVENTS, FORMS AND SPACES

1.1.1. Phenomenological aspects of sound events and sound forms

Light and sound are the two windows which allow beings to come into direct contact with the universe. From the wide range of existing wave and corpuscular phenomena, beings can feel and translate into sensations only narrow strips, which become their principal instruments in world investigation and understanding.

As a person living inside a tower knows about the surrounding landscape only what may be seen through two slits in the thick walls, beings are acquainted with the universe through these two windows open to the specific wave and corpuscular phenomena generating light and sound sensations.

A wave phenomenon of audible frequency is perceived and reflected in one's consciousness as a ***sound event.***

A sound event is a subjective fact, a mental condition; it exists only as a sensation, a reality inside beings. There is an isomorphic relationship between the objective parameter values of a "sound source" and the subjective, inner object they create. Duration, amplitude and compound frequencies of oscillations of wave phenomenon are grasped by one's consciousness as characteristics of a sound event: time lengths, loudness, pitch, timbre...

A sound event is the opposite of a "non-sound event" that is of stillness, of quietness. Sound events arise as "islands" in an ocean of silence.

Several sound events may constitute in consciousness a distinct unit, a *sound form*. The sound form may be interpreted as a set whose elements are sound events.

The concept of *form* – as an object of the visual domain – has been present in philosophical thought since ancient times. Starting with Plato, almost all philosophers have been attracted by the problem of form. However, practically exclusively, they took interest only in the *visual form.*

According to Aristotle, *form* means the principles of organization and unity of every thing or being. For Kant, *form* represents the laws of thinking that establish relationships between data transferred through sensations. Philosophers started gradually to grasp the way in which the relationships existing between a number of elements constitute a unit. In the 20th century, the *theory of forms* (*Gestalttheorie*) provided an important step in this direction. This theory points out that the characteristics of phenomena do not result from adding up the properties of their constitutive elements, but from the aggregate effect of the interrelationships of the considered elements. Contemporary structuralism has developed all these ideas and has applied them to various fields in order to study the interdependence of the component factors in complex phenomena.

In the context of the present work the concept of **sound form** – as an object of the auditory domain – implies as well the idea of relationships between its constitutive elements. The brain unceasingly compares the perceived sound events, relating them to each other from the point of view of their appearance, disappearance, pitch, timbre, intensity, direction, etc. In other words, the relationships between the elements of sound forms are established by comparing the values of each of the specific parameters of the considered sounds.

Memory plays an essential role in perceiving sound forms. In order to obtain a global sensation and to enable the impression of sound form to be created, the memory stores the sound events constituting the sound form, while they unfold. The association of these events by means of memory leads to their overall perception, where the whole acquires a distinct significance, superior to the union of the component elements.

Certain characteristics of sound forms differentiate them fundamentally from visual forms. Here are two of the most important ones:

(a) While visual forms are perceived by means of light, the wave and corpuscular phenomenon being an instrument in the perception of the *form*, the *sound forms* are a product of subjective consciousness of some specific wave phenomena and have no objective existence outside it. The *sound forms* cannot be "felt" or "touched".

b) In the main, visual forms are instantly perceived as a whole, the global impression being established the very moment the object is seen, while the sound forms – as time phenomena – unfold and are grasped in time. The *sound form* can be registered as a whole by consciousness only at the end of the time interval over which it extends.

1.1.2. Definition

By **sound form** we understand the set of relationships established by consciousness among a finite number of sound events.

"The deep silence was broken by a near shrill whistle, followed after a short time by a far-off muffled noise of a collapsing body, then silence came again..."

In the above sentence, somebody actually defines a sound form by establishing relationships among a number of sound events. The person's consciousness makes an order relation:

– in time, depending on the appearing and disappearing moments of the whistle and the collapsing body's noise;
– on the frequency scale, depending on the pitch of these events;
– on the intensity scale, depending on the sound intensity;
– on an arbitrary and subjective scale of timbres;
– on a scale of sound source distances and positions in space.

It is easy to see that there exists a bijective relationship between the objective phenomena of the oscillations set forth by physical bodies and the subjective image of these oscillations perceived in one's consciousness as "*sound forms*". Also, it may be noticed that there is a bicontinuous relationship between the unfolding in time of the oscillations and the settling in time of the sensation of *sound form* in consciousness. *Consequently, we may state that there is a topological correspondence between the sound form perception and the action of sound sources generating it.*

1.1.3. Sound space. Formalization of the concepts of sound point, sound event and sound form. Null and imaginary sound events

In the following, we will formalize the concepts of sound event and sound form and turn the sound structure into a space structure, using the space structure not only as a code, but also as a mode to understand and develop the sound structure.

While examining a sound event, which is, as already seen, the reflection in consciousness of a wave phenomenon aurally perceived, four sound parameters are at first noticeable: (a) duration, (b) pitch, (c) intensity, and (d) timbre.

The mode of attack (e) (the sound transient zone characteristic feature) and the position in the Euclidean space of the sound source defined in its

turn by three values corresponding to the space coordinates (*f, g, h*) may also be added to the four above mentioned parameters.

In order to apply a mathematical instrument to the study of *sound forms* we will draw up a list of definitions, as follows:

Sound space

Sound space is an 8-dimension topological space, having as space coordinates the *a,b...h* sound parameters considered.

The *sound space* appears as a Cartesian product $A * B * C * ... * H$, where *A* is the set of values covered by *a*, *B* the set of values covered by *b*, etc. The *sound space* is the domain of sound events.

A *sound point* is a sound event defined by only one point $(a, b, c, ..., h)$ in the set $A * B * C * ... * H$. The *sound point* is the simplest *sound event.*

A connected part of the Cartesian product of some finite intervals specified in the $A * B * C * ... * H$ set is a *sound event.*

The *sound event* may be imagined as a part of a parallelepiped whose *h* dimensions are sound parameters.

Since any connected set of *sound points* is a *sound event* (according to the previous definition), a non-connected set of *sound points* will be a set of *sound events.*

Null sound event

Let *a, b* and *c* denote sound parameters: duration, pitch, and intensity, respectively. A sound event is *null* when, at moment t_x or in a (t_{xp}, t_{xq}) time interval, the value of at least one of the considered parameters is zero. For instance, the vanishing of factor *a* leads to a sound event generated by a wave phenomenon of zero duration, the vanishing of factor *b* leads to a sound event generated by a wave phenomenon of zero oscillations frequency, the vanishing of factor *c* leads to a sound event generated by a wave phenomenon whose amplitude is zero. The sound point is a null sound event since, by definition, its duration is zero.

Imaginary sound event

A sound event is imaginary when the values of one of its pitch or intensity parameter exceed or are below the perceptibility threshold. For instance, let *A* be the set of sound pitch parameter values. We know that, generally, human ear cannot hear sounds with frequencies lower than approx. 16 Hz or higher than approx. 25,000 Hz. When the frequency of a wave phenomenon generating a *sound event* is situated outside the above-mentioned interval, the respective event is an imaginary *sound event.* Likewise, a *sound event* is imaginary when the amplitude of a wave phenomenon generating it is below the audibility threshold.

Some additional definitions

A sound event is a sound form (a set with only one element).

The sound point is the simplest sound form.

The sets of connected points representing non-null sound events in the sound space correspond in our consciousness to the sensation of sounds events. The sets of points representing null sound events (1.1.3b), and the complement of the union of the sets of non-null sound events correspond to the sensation of silence.

Consciousness and mathematical model of sound form

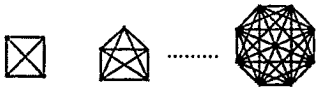
The definition of the *sound form* as being the set of relationships established in a finite set of *sound events* (1.1.2) may be interpreted as the projection in our consciousness of the Cartesian product of a family of non-null *sound events.*

Indeed, the way consciousness relates and compares a finite number of *sound events* is nothing other than the comparison of parameter values for the respective *sound events* at a given moment, or while surveying the evolution of the established relationships throughout a time interval.

From a mathematical point of view these relationships correspond to some parts of a number of Cartesian products of finite intervals defined on the coordinates axes, each Cartesian product representing the operation of comparing the parameter values of the sound events under consideration.

The number *r* of relationships that may be established among a number *p* of elements – sound events – is given by the formula $r = p(p - 1)/2$, i.e., only one relation may be established between two elements, three relations among three elements, six among four elements and ten among five, etc. Fig. 1.1 illustrates this fact. The elements under consideration are points in a geometrical space.

Fig. 1.1



Consequently, r Cartesian products may be established among a number p of sound events (imaginable as parts of some n-dimensional parallelepipeds). Each such r Cartesian product represents the operation of comparing the parameter values of a couple of events from the p events under consideration. These r Cartesian products may be again structured as Cartesian product and imagined as an m-dimensional parallelepiped. *A part of this m-dimensional parallelepiped represents, from a mathematical point of view, the sound form as a consciousness process.*

1.1.4. Space representation of the sound form

A space representation of the concept of sound form as part of an l-dimensional parallelepiped, i.e. as a Cartesian product of sets of sound events, defined, in their turn, as parts of some products of finite value intervals of some sound parameters, is obviously impossible, as one cannot draw on paper mathematical objects of more than three dimensions.

However, we will try to guide the reader towards the understanding of the mathematical model of the consciousness phenomenon of *sound form* by using – as far as possible – a graphical representation.

Let us start by representing three sound events E_1, E_2 and E_3 in the sound space $\mathbf{R}^2$ having as dimensions time (t) and sound pitch (p).

Each E_I event (I = 1, 2, 3) may be considered as part of the Cartesian product of the respective intervals T_i and I_i (Fig. 1.2). The choice of the origin of the coordinates is arbitrary.

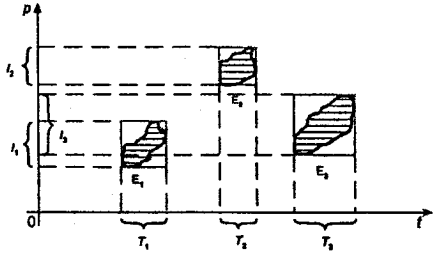


Fig. 1.2

The Cartesian products E_1 * E_2, E_2 * E_3 and E_1 * E_3 represent the consciousness operations when comparing, by couples, the parameter values of the respective events.

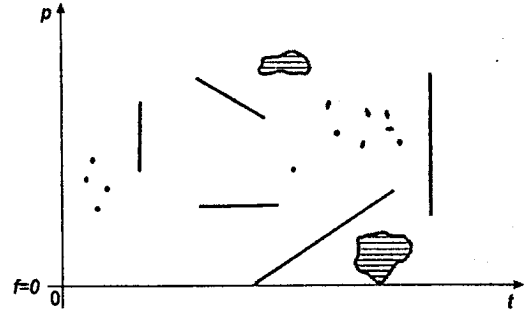
Let P_1, P_2 and P_3 be these products.

Since each element E_i is a part of a Cartesian product of two finite intervals, the result is that each product P_i may be imagined as a part of a four-

dimensional parallelepiped (2 * 2). The three P products may be structured again as a Cartesian product P_1 * P_2 * P_3, which can be considered with the mind's eye as a part of a 64-dimensional parallelepiped...

Now, let us consider a three-dimensional sound space $\mathbf{R}^3$ whose dimensions are sound parameters: time ($0x$), pitch ($0y$) and intensity ($0z$). The values on the 0x dimension are expressed in decibels.The three events will have the form of a part of a parallelepiped in this three-dimensional space (Fig. 1.3).

Fig. 1.3



By proceeding as in the previous example, we will first obtain three parts of 9-dimensional parallelepipeds, which, when structured again as a Cartesian product, may represent the sound form as a part of a parallelepiped "living" in a space of 729 dimensions!

Under the above circumstances, we have considered a very simple sound form of only three elements. But, as we will see, musical sounds are sound events; the perception of a symphony does not mean registering sounds in our consciousness but the relationships among their duration, pitch, intensity and timbre. We urge the reader to close the book for a moment and ponder upon his preferred symphony, considered from the angle of the set products called after the great French philosopher and mathematician Descartes and contemplate the sound universe of music in the spirit of the verses from the Upanishad's ADVAYA-TARAKA. *The essence of vision is only possible through the mind's eye...*

1.1.5. Real sound space. Representation of sound forms in the real sound space

The purpose of this work is the study of musical sound structures and their generation. Since the tradition of musical treatises limits the study to sound pitch and duration and, since too many parameters would cause difficulties to the investigation and generating process, in the following we

will limit ourselves to the two fundamental dimensions of the musical sound: pitch and duration.

In order to simplify the reasoning it is more advantageous to work with the sound form not as a Cartesian product, the way it is perceived by human consciousness, but with the family of the sound event itself, represented as connected sets of sound points in the space defined by sets *A* and *B* (duration and pitch values). The sound space thus defined is referred to as ***Real Sound Space.*** (In the following we will introduce the concept of ***Virtual Sound Space.***) The *Real Sound Space* is a topological space $\mathbf{R}^2$ with a vectorial structure (A.2.1.), time (*t*) and sound pitch (*p*) being the space coordinates.

Real Sound Events are sound events represented in this space.

Musical sounds are sound events represented in the sound space $\mathbf{R}^2$, as straight line segments, in general parallel to the time axis (sounds of constant pitch), sometimes as curve segments.

The null sound events occur as straight line segments perpendicular to the time axis (sound events of zero duration), as straight line segments perpendicular to the pitch axis representing a sound event produced by a sound source of oscillations of zero frequency, the boundary of some connected sound points sets reaching null oscillation frequency, as well as isolated points in sound space $\mathbf{R}^2$ (Fig. 1.4).

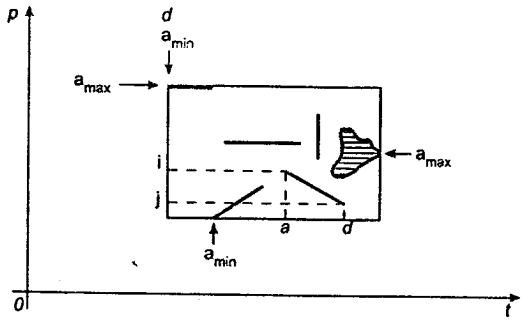


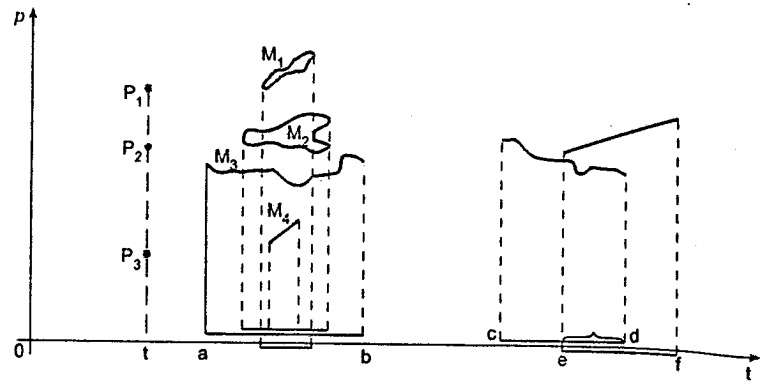
Fig. 1.4

Musical sounds are sound events represented in the sound space **R2** as straight line segments, in general parallel to the time axis (sounds of constant pitch), sometimes as curve segments.

1.1.6. Sound form space. Dimensions of the sound event and of the sound form

The space of a sound form is the space within the smallest rectangle with sides parallel to the axes containing all the sound events belonging to the given sound form (Fig. 1.5). The sound form space is a convex space.

Fig. 1.5



The dimensions of a sound event are given by the following relations:

(1) $d - a$, where a is the abscissa of the moment of the event appearance and d the abscissa of its moment of disappearance;

(2) $I - j$, where i is the ordinate of the highest pitch and j the ordinate of the lowest pitch of the considered sound event.

A sound event duration is given by relationship (1).

The ambitus[1] of a sound event is the value given by relationship (2).

In a similar way, the duration and the ambitus of a sound form can be determined.

When relations (1) and (2) have zero values, a sound event dimension is reduced to a sound point.

The set of points defined by the convex space of the sound form and by the null and non-null sound events, represented as subsets of this space, defines a topology on the sound form with the specification that, from the aural point of view, the union and intersection of point sets representing non-null sound events occur when their abscissae have the same values) (Fig. 1.6).

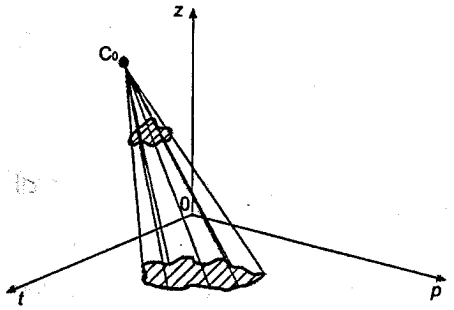


Fig. 1.6

[1] "Ambitus" is used in music to define the interval between the minimum and the maximum pitch values of a musical fragment.

The sets of points representing non-null sound events are open sets in the usual topology of the convex space, as part of the real sound space; the sets of points representing null sound events are closed sets, complements of open sets of this space.

1.1.7. Virtual sound space. Virtual sound forms

In order to obtain the largest possible variety of homomorphic images of a sound (A.2.2), the real sound space $\mathbf{R}^2$ is associated with an operational space $\mathbf{R}^3$ referred to as *virtual sound space*. Two of the dimensions of this space are the dimensions of the real sound space $\mathbf{R}^2$, and the third is an abstract algebraical dimension. The sound forms represented in the virtual sound space $\mathbf{R}^3$ are *virtual sound forms* having *virtual sound events* as elements. The virtual sound forms are transformed within $\mathbf{R}^3$ and projected in the real sound space $\mathbf{R}^2$ (A.3.20) (Fig. 1.7).

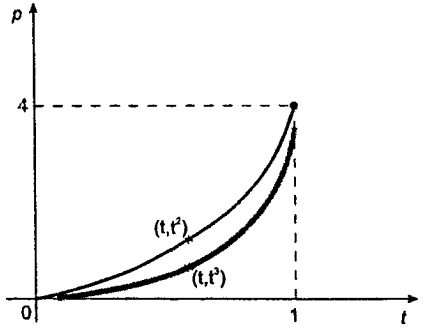


Fig. 1.7

1.2. SOUND FORMS AND TOPOLOGICAL TRANSFORMATIONS

1.2.1. Sound forms and the concept of topological transformation

A special case of topological space is represented by the sound form as specified under (1.1.2). The convex space of the form in $\mathbf{R}^2$ is the space support, and the topology is the one induced by the space $\mathbf{R}^2$ topology.

Now, we will extend the concept of sound form by generalizing it to the set of homomorphic elements obtained by a group of topological transformations.

Let E and E' be two sets of sound events, consisting of the sets of sound points p, q, r, and p',q',r', ... E' is the topological transformation of E through $(E' = T(E))$ when each sound point in E' is the transformed image of one and only one point from E (i.e., $p' = T(p)$ and $q' = T(q)$, with $p \neq q$, implying $p' \neq q'$) and reciprocally, when each sound point in E is the transformed image of one, and only one point from E' (i.e., $p = T^{-1}(p')$ and $q = T^{-1}(q')$ with $p' \neq q'$ implying that $p \neq q$) and, in addition, to every neighborhood of the sound point p in E, there is a corresponding neighborhood of the sound point p' in E' and reciprocally, for each neighborhood of the sound point p' in E' there is a corresponding neighborhood of the point p in E through $^{-1}$.

The set of topological transformations forms a group.

We recall that a binary relationship in E is, by definition, an equivalence relationship when it is reflexive, symmetrical and transitive, and that an equivalence class is a subset in E which includes all the sound events, elements of E, equivalent to a given element, e.

The concept of topological transformation and the concept of equivalence relationship are interconnected by the following property:

The relationship between a set E and its topological transformation is an equivalence relationship.

This property may be also expressed as: all the transformed images of a set through a group transformations form an equivalence class.

Starting from the fact that a figure is a set of points in a plane and that several figures, interconnected through the congruence or similarity relationship, are considered as having the same form, and knowing that equality and similarity relationships are equivalence relationships, we may say that the form of a figure is the equivalence class of all the congruent or similar figures with a given figure, the class being considered as an element of the quotient set. In a more general way, we may say that: *The form of a figure with respect to an equivalence relationship is the equivalence class consisting of all the figures obtained from the given figure by the considered equivalence relationship.*

Since we know that there is an isomorphism between a figure interpreted as a set of points in a plane and a set of sound events, represented by sets of sound points in the sound space, we will be able to give a new and more general definition of the sound form related to a group of transformations.

Definition 1

The sound form is the class of equivalence consisting of the family of sound events sets, obtained from a set of given sound events, through a group of transformations.

By considering Klein's definition, where geometry is the study of the properties of the figures that are invariant under the transformations of a group, and knowing that the group of topological transformations we will operate with on the sound forms includes some fundamental geometrical transformations, we may agree upon the following:

Definition 2

Topology of sound forms is the study of sound forms with respect to the group of topological transformations.

1.2.2. Musical sounds: curves in the sound space $\mathbf{R}^2$

In the following we will direct our interest especially to the curves in $\mathbf{R}^2$, interpreted as time-pitch bidimensional sound space. The curves in this space actually define the sounds since the sound concept, as defined in (1.1.6), implies a linearity of the sound event under consideration, a certain linear order of the sound points forming it. The linear order in a time interval $[a, b]$ introduces, through the agency of the vectorial mapping f, a linear order of the points of the curve. Since the [a, b] interval is always situated on the time axis, as a matter of fact, we are dealing with a special class of curves in $\mathbf{R}^2$, time itself being the parameter. The parametric representation of these curves is therefore of the form:

$$x = t, \quad y = f(t), \quad a < t < b$$

where f is a continuous function on the $[a, b]$ interval on the time axis.

The curve topological transformations, as defined up to now, referred to the curve functional hypostasis, when the curve is a specific mapping (A.1.1). It will be necessary in the following to refer also to the curve images and, in this sense, we will define a new concept allowing the respective curves to have different images.

Two curves C: $[a,b] \to \mathbf{R}^n$ and C': $[a',b'] \to \mathbf{R}^n$ are topologically equivalent if $\exists$ h: $[a,b] \to [a',b']$ homomorphism such that $\forall\ t \in [a,b]$ C' (t) = = C(h(t))...

The following is a simple example: Let C_1 be given by $x = t$, $y = 1$, $(0 \le t \le 1)$ and C_2 be given by $x = t$, $y = 2$, $(0 \le t \le 1)$. We can prove that C_1 and C_2 are topologically equivalent in the weak sense. Indeed, let us

consider the homomorphism h between the straight line segments Γ_1 and Γ_2 which are the graphs of C_1 and C_2, the homomorphism being defined as follows: when the point P_1 belongs ton Γ_1, we associate through h on Γ_2 the point P_2 of the same abscissa as P_1.

Topological transformations in the weak sense of the plane curves allow us to define the corresponding sound transformations, as the sounds are represented precisely by curves in the $\mathbf{R}^2$ sound space. Thus, the topological transformations in the weak sense enable an interpretation of these curves, considered as representing sounds. An example of such a transformation is that shown above of the curves C_1 and C_2, having as image some line segments. Below is an example a little more complicated. Let us consider the following curves:

$$C_1 = \begin{cases} x = t \\ y = t^2 \end{cases} \qquad C_2 = \begin{cases} x = t \\ y = t^3 \end{cases} \qquad 0 \le t \le 1$$

We will define the homomorphism h by the bijection which associates to the point of coordinates (t, t^2) from the image of C_1, the point of coordinates (t, t^3) from the image of C_2. The correspondence is given in Fig. 1.8:

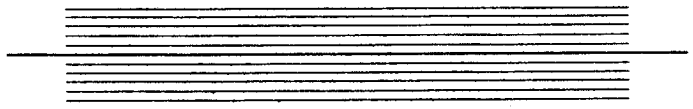


Fig. 1.8

When we interpret the abscissa as time and the ordinate as sound pitch, C_1 and C_2 define two sounds S_1 and S_2 in the sound space, having the same moment of appearance and of disappearance.

In the example surveyed, graphical representation of the sounds was possible as they have been defined by means of mappings, not only continuous, but also having a continuous derivative. As regards the sounds represented by continuous mappings which are not derivable, or by derivable mappings having discontinuous derivative in a dense set of points, they can no longer be plotted; hence, the only mode of operation with these sounds is the analytical one, any geometrical intuition becoming impossible. This is not an impediment to their examination and to their topological transformation in the weak sense using mathematical analysis and the theory of calculable functions. The "calculable" sounds and the "calculable" transformations may be obtained irrespective of the possibility of their

geometrical representation. The sounds thus obtained, even when they occur as imaginary sounds, may be converted – by transposition – into the audible field.

1.3. FROM THE GENERAL SOUND EVENT TO SPOKEN LANGUAGES AND MUSICAL STRUCTURES

1.3.1. Classification of sound forms: signs and signals

Recalling from the Information Theory the concepts of *signs* and *signals* and applying them to the concept of sound form we may say:

1. by *sound forms – signs,* are understood sound forms associated to some phenomena as a secondary result of their generation, e.g.:

(a) sound forms generated by certain natural phenomena: thunder, waterfall, the falling of a tree, etc.;

(b) sound forms generated by the physical activity of beings: fluttering of wings, an animal treading over leaves, noise of a hammer handled by a worker, etc.;

(c) sound forms associated to certain emotional states as: screams of pain, weeping, laughter, sighing, etc.;

(d) sound forms generated by different devices as a result of their activity: the noise of the motor of a car, the ticking of a grandfather clock, etc.;

2. by *sound forms – signals,* are understood sound forms as the result of certain physical phenomena generated to carry a message, e.g.:

(a) sound languages of living creatures: barking, twittering, and, of course, natural human spoken languages;

(b) different sounds mechanically or electronically created in order to produce signaling with a language character: sirens for factories and ships, Morse sound signals, etc.;

(c) sound forms as a result of some sound means of artistic expression, the so-called artistic sound languages, first of all Music.

In the category of signal sound forms, foremost place is held by spoken human languages as a special case of sound form requiring a more extensive discussion.

1.3.2. The role of sound forms in natural spoken languages. Two modes of transmitting information in spoken languages: the logic-rational language and the emotional language. Code and resonance

By speech, communication is made by means of specific sound forms, the words. Words are completely ordered finite sequences generated through arrangements including repetition of some basic elements, i.e. the set of sounds of a given language. The combinatorial operations on the obtained sequences lead to the generation of sentences. Some elementary notions for the generation of words and sentences are to be found under (1.3.3).

In the process of expressing ideas and sentiments, we differentiate two types of spoken languages which we will refer to as *logic-rational language* and *psycho-emotional language*. The logic-rational language carries concepts and relationships between concepts, rationally understood by the receiver, and decoded by a thinking process. The psycho-emotional language carries sentiments, emotions and sensations which are "felt" by the receiver, caught at the consciousness affective level through a process of entering into *psychic resonance.*

For instance, in rational language, in order to transmit information about temperature, it is enough to utter sounds representing the sound symbol of a figure, the number of degrees. Rationally, the interlocutor will be perfectly informed.

In emotional language, a scream of pain due to a violent sensation for example of burning, will generate emotion in the persons hearing it. They will *feel* intuitively the sensation of pain, directly perceiving it by *entering into resonance.*

Strictly referring to the psycho-emotional language, we will be able to determine two modes of transmitting the information carried by this specific language:

1. by the connotative property of terms, i.e., their capacity to express, apart from their usual meanings, some secondary ones obtained with poetical, metaphorical figures. An extensive study of this aspect of the language has been carried out by Soloman Marcus in his "Mathematische Poetik";[2]

2. by the expressiveness contained in the mode of uttering and of the pronunciation of the word, i.e., by its *sound form.* We will refer to this language aspect as psycho-emotional sound language.

[2] Solomon Marcus, *Mathematische Poetik,* Atheneum Verlag, Frankfurt am Main, 1973.

The importance of the sound aspect of the spoken language in transmitting information has been frequently emphasized. Thus, Eugen Lovinescu[3] says that "besides its value as a conventional sign, any word is also a store of many potential meanings and suggestions *due to its sound quality*" (our underlining). In the same sense, Noël Mouloud[4] states: "C'est-à-dire que syntaxe et sémantique ne suffisent pas a définir la structure du langage: celui-ci redevient "forme" dans un sens plus "concret" et plus "primitif" du terme: il s'appuie directement sur *l'expression spatiale et temporelle de ses "tonalités"* (our underlining).

Transmitting emotions and not describing them, aiming to bilaterally establish an emotion–emotion connection not by emotion/reason–reason/emotion couples, the sound psycho-emotional language is situated in the consciousness infralogical domain within the area where the sound "expression" becomes the direct means of communication of the emotive information.

"Purely" emotional language is, as a rule, restrained and its signals carry sensations in a non-differentiated, global mode: a scream transmits a violent sensation, possibly its intensity also, but not its quality. A scream could mean pain through hitting, but also through burning or stinging; the same scream could, however, manifest horror, fright, reaction in front of a chasm or in front of a blinding light.

"The purely emotional in language is represented by interjections. These are moving far from the referential language procedures both by their phonic configuration (they include special phonic sequences, even sounds unusual elsewhere), and also by their synthetic role (an interjection is not an element of a sentence, but the equivalent of a complete sentence)", remarks Roman Jakobson.[5]

The sound range of weeping and laughter, sighing, screaming, howling, whispering, sobbing, etc., involves specific sound forms. We have pointed out that these sound forms accompany and convey information about certain emotional conditions. While in logic-rational language sound forms occur for information and concept transmission only, those uttered during certain emotional states are an integral part, components of these emotional states, and manifest themselves involuntarily. This is the case even when a person experiencing heightened emotions is alone or does not want to inform others about his condition or sensations, and even more so, when he desires to hide them.

[3] Eugen Lovinescu, *Istoria literaturii române contemporane*, vol. I, Ed. Socec, 1937.
[4] Noël Mouloud, *Pictura şi spaţiul*, Ed. Meridiane, Bucureşti, 1978.
[5] Roman Jakobson, *Essais de linguistique générale*, Paris, 1963, p. 214.

We may state briefly that:

1. The psycho-emotional sound language does not convey sound-coded concepts. It directly determines emotional states. These emotional states are entries into resonance.

2. While, in logic-rational language the main element is the assembling mode of sounds – the vowels and consonants becoming words, "significants" of certain concepts – in emotional language the sound modalities of emission occur as dominants: pitch, timbre, intensity.

1.3.3. Emotional modulation of the logic-rational spoken language

In ordinary speech, normally logic-rational language occurs very seldom in a "pure" state. Even in a sentence including an objective statement, the speaker adds to the words an "emotional modulation" which is obtained by means of the "expression" he gives to the sentence, by the mode of uttering the words i.e., by the sound intensity (whisper, violent speech), sound pitch (acute scream, murmur), and speed of pronouncing the words (excited or calm speech).

To the pure information he is conveying, the speaker adds his own emotional stance relative to what he is saying. From the city dweller's displeasure at having to give up his walk to the joy of a farmer whose crop is saved, there are endless manners of saying "it is raining." In addition to the pure information, the words "it is raining" contain information that we do understand, but the way these words are pronounced, "their emotional modulation," enable us to feel the informer's standpoint. Hence, on reception, the receiving person rationally "decodes" the information that the sounds are transmitting and "gets into resonance," i.e., feels the emotional message held in the "modulation" of the respective sounds. The so-called "expression" is thus included in the way sounds are emitted and through the values of their parameters, intensity, pitch, duration, timbre.

In certain cases the emotional information can undergo a transfer into the rational field. The reiterated reception of certain sound signals, which are purely emotional, for instance, a certain manner in which a sick person sighs, this sound primarily received by resonance, becomes a code. Somebody who has repeatedly received it, in the same form, will rationally understand that the person is suffering and needs help, "Sigh" has become a "Word".

1.3.4. Combinatorial and parametric aspects in spoken languages

In spoken natural languages, information is conveyed by sound forms whose elements are the words. We may point out two different aspects in the "spoken word" sound event structure:

a. an aspect obtained by combinatorial systems applied to some finite sets of fundamental sound events referred to as sounds of a language;

b. an aspect represented by a connected part of the Cartesian product of the sets of points representing the values of sound parameters involved (1.1.3,4).

We will refer to aspect (1) as the combinatorial aspect and to aspect (2) as the parametric aspect of the spoken language. An infinity of parametric aspects may be associated in practice to a word defined through its combinatorial aspect. By exception the word connotative senses (metaphors, etc.) carrying emotional meanings are expressed through their combinatorial aspect.

1.3.5. Combinatorial systems and topological transformations used for transmitting information through sound forms. Combinatorial systems of word generation

What follows will provide some elementary concepts illustrating the mode of generating words and different operations with them.

The combinatorial system of generating words and sentences, hence the conveying of logic-rational information carried by the language, have been thoroughly examined in mathematical logic as combinatorial systems, associative calculations and generative grammars. Other mathematical procedures for these are the Turing machines, recursive functions and different types of automata. All these systems have the ability to generate sets of sequences through certain rules. Generation of sound forms in natural spoken languages is also a mechanism of this type.

Considering that operations regarding the transmission of logic-rational information in a natural spoken language deal only with strictly denotative senses of words, hence with none of the derived senses, metaphors, allegories, etc. (in other words, only one sense corresponds to each word), the study of word generation in such a language coincides with the study of word generation through operations on the "written form"[6] of sounds.

[6] The mode of uttering the words is not relevant; no "emotional modulation" of the sounds is involved in this process.

The combinatorial system (1) operates in two phases:

(a) generation of completely ordered finite sequences through arrangements (that may include repetition) of the set of sounds of the given language;

(b) operations on the sequences obtained, intended to create new ones by the union of two or more given sentences or to bring changes in the order occurrence of the sounds.

Sequences and their concatenation

Let A be a set consisting of three elements: a, b, c.
Then

$b\ a\ c\ a\ b\ a$

will represent a "sequence" of six terms, each one being an element of A. For a given sequence $s = a_1, a_2, \ldots a_n$, the number n is called the sequence length, and it is denoted by \s\. We have thus \b a c a b a\ = 6. We also introduce the notion of *null-sequence*, i.e., that sequence containing no element of A. The null-sequence corresponds to the space (blank) between the words.

If B and C are two given sequences, a correspondence can be defined between the ordered couple (B, C) and a sequence D, obtained by concatenation of B and C.

$D = BC$
B = "baca", C = "ac", BC = "bacaac"
CB = "acbaca"

Associative calculation

Let A, B, P and Q be four sequences. If, in the sequence APB we put Q instead of P, we obtain the sequence AQB.

Assuming that both sequences APB and AQB have sense (the nature of this sense varying in each case), we say that P substitution by Q in APB is possible. Thus, we can take A = con, B = tion, P = secra and Q = secu, and we obtain APB = consecration, and AQB = consecution.

A finite set of non-null and null sequences arranged according to some rules forms a phrase.

1.3.6. Topological space structure of sound forms in spoken languages

We have defined sound events as parts of the Cartesian product of some finite intervals of sound parameter values (1.1.3).

The sets $A...G$ are of the power of continuum.

In the set of sets whose product defines the "words" events in spoken languages, there also occurs a set @ (sounds of the respective language) which is finite (numerable, at the most). The occurrence of this finite set admits a deeper and more specific structure in the topological space of sound forms: the sound space of spoken languages.

1.3.7. Topological transformations mapped to sound forms representing words and phrases. Artistic interpretation of a text considered as a topological transformation of the respective text sound form

We have mentioned that a sound event is the projection in our consciousness of a connected part of the Cartesian product of sets representing sound parameter values defining it and that there is a topological correspondence between the perception of the sound events and the action of the sound sources generating them. We have also shown that sound events may be interpreted in the sound space $\mathbf{R}^{n7}$ as connected sets of sound points and that, when through transformation such a set keeps its quality of being connected, and if between the object and its obtained image, there is a biunivoque and bicontinuous relationship, the transformation will then be a topological one.

The sound event, representing the word, is globally perceived and not as a sequence of separate adjoining sounds. Within the sound language field, a sequence of uttered sounds is different from the pronunciation of the sequence of sounds forming it, e.g., call ≠ c.a.l.l.

By "*emotional modulation*" one changes the values of sound parameters defining the word-sound event, this keeping its connectedness at a moment of perception and in its spatial representation, as well as the order of the sound points forming it. Also the respective mapping is bijective and bicontinuous.

What is valid for the sound event as element of a sound form is also valid for the sound form containing several word-elements: a phrase.

Consequently, we may formulate the following **Remark:**

In spoken languages "Emotional Modulation" denotes an information transformation carried out by means of topological transformations applied to the sound form representing a word or a phrase. This transformation is obtained by changing the parametric aspect of the considered sound form, i.e., by modifying the values of parameters defining the Cartesian product representing it (1.1.3).

[7] *n* represents the number of the considered sound parameters.

Since the images obtained through a topological transformation of a sound form constitute an equivalence class (1.2.1) and the set of class elements is of the power of continuum, this means that the number of ways of expressing a word or a phrase is practically infinite. It is possible to associate to a word (considered as an element of a finite set of sequences obtained through combinatorial system applied to the set of sounds of a language) a practically infinite number of parametric aspects, each aspect carrying a certain sense, a certain shade, a certain psychological state. This explains why the same word sounds different when pronounced by different persons, why a text unexpressively read may become full of meanings when uttered by a great actor, why in theater the same part, dull and tedious when played by a second-rate performer, can become fascinating when presented by a gifted artist.

Consequently, we can state that the artistic interpretation of a text is obtained through topological transformations applied to the sound forms represented by the considered text.

1.3.8. Graphical aspect of natural languages. Writing: a filter of information conveyed through topological transformations applied to the sound forms of spoken languages

Before continuing, let us recall the difference between the logic-rational aspect and the emotional one in spoken language. The former, which we referred to as logic-rational language, transmits concepts and concept interrelations; the latter, the psycho-emotional language, conveys sensations and emotional states.

The logic-rational language is addressed to the mind acting rationally; the psycho-emotional one is addressed to the emotional domain and acts in infralogical areas. The logic-rational language conveys rational information, concepts coded in sounds to be "decoded," rationally understood; the psycho-emotional language conveys emotional information directly determining emotional states to the interlocutor, perceived through a phenomenon similar to the "resonance" in acoustics.

In the spoken language the logic-rational information is conveyed through the combinatorial aspect of the words, i.e., through the arrangement of sounds of the respective language, while the psycho-emotional information is conveyed through the parametric aspect of words (1.3.4).

This explains why the logic-rational language can be perfectly written making use of the signs of the alphabet, while the writing of the psycho-emotional language is problematic, because it supposes the graphical representation of the parametric aspect of sounds which cannot be "written" with the signs of the alphabet.

The sounds uttered in certain emotional states can almost never find their sound correspondent in the sounds of any language. Since other means do not exist, some sounds of the respective language, which seems to be the nearest to those uttered in certain emotional states, are used to write them. In other words, we use the signs of the rational language in order to graphically express some sound events determined by certain emotional states. Of course, this method is not adequate, and we know how difficult it is to "write" laughter, weeping, howling or sighing by means of usual phonetic signs – i.e, with the alphabet letters.

Intonation, intensity, rhythm and speed of words uttered in spoken languages "are lost" when writing them down. The act of reciting a poetical text, by which the interpreter "modulates" the text, trying to give it the highest emotional efficiency, cannot be graphically "captured," "solidified".

The verbal communication, emotionally loaded by rhythm, intonation, variation of intensity and word uttering speed, is "stripped," "deprived" of the emotional content expressed by the sound parameters.

Recalling that in spoken languages the emotional information is carried through topological transformations applied to the sound forms representing the words (1.3.7) we may put down the following **Remark:**

The writing is a "filter" for the emotional information expressed through topological transformations applied to the sound forms of a spoken language.

1.4. THE ROLE OF SOUND FORMS IN THE MUSICAL LANGUAGE

1.4.1. A higher degree of text emotional modulation: the song

In a song, the emotional modulation of a word sequence is given in a more obvious and precisely organized form. From the set of all possible readings obtained by varying the pitch and the duration of the syllables-sounds, the author-composer will choose the ones he prefers. Syllables-sounds of the word sequence therefore become quite different from each other as pitch and duration, alternating with silence zones which correspond to the null sequences.

The song appears therefore as a higher step of emotional modulation of a given text. We could assume that, historically, the song appeared as a consequence of the emotional modulation phenomenon of speech. One of the most typical examples, and maybe one of the most beautiful ones, is the Romanian "*Doina*"; this is an essentially melancholic song where the words are emotionally modulated by a melodic structure.

As a conclusion we may assert that in "song language," certain well defined pitch values and duration are associated to syllables forming words.

1.4.2. Musical "modes": "containers" of finite sets of pitch values

Unlike the spoken language, where sound pitch may take continuous values in the audible frequency band, in a song the pitches of the syllable are selected from a finite set of pitches, a "container" holding musical sounds pertaining to specific musical "modes" (2.2.1).

1.4.3. A "container" holding a finite set of musical duration

In folk song, the value of the sounds and rests duration cannot always be rigorously integrated in the known duration system of the traditional theory of music where they are obtained by dividing successively a fundamental time unit in various modes. But, since in studies of folk music, the sound duration associated with syllables are (sometimes improperly) written down with the usual music symbols, we might say that in the "song sound form" the musical duration (whole note, quarter note, eighth note, etc.) constitute a finite set from which the author (often anonymous) builds his melody, as far as the time parameter is concerned.

1.4.4. Song sound space structure

As mentioned above in the case of natural spoken languages, the song may be interpreted as a sound form whose elements, sound events, "song words," are parts of the Cartesian product of the sets @ (language sounds) and the sets of sound parameter values defining the sound space, field of the

Unlike the spoken language space, where, excepting @ (finite), the sets *A, B, C*, and *D*, are of the power of continuum, the *song sound space* has a deeper structure, more precisely defined, as the sets *A* and *B* are also finite or numerable, at most. The sets *C* and *D* remain of the power of continuum, since, in principle, a song may be sung with an infinity of intensity values, and the human voice may actually have an infinite number of timbres.

1.4.5. The act of performing a song: topological transformation of the sound form representing the considered song

Like in spoken language, the parametric aspect of sound forms representing the song may be topologically transformed by changing the sound's unfolding speed (modification of tempo, thus expanding or compressing the duration of the sounds and rests) by transposing the sound form on the pitch scale (transposition), thus expanding or compressing some empty sequences, by changing the intensity, the timbre, etc. The above changes take place, as a rule, in what we refer to as "the act of performance", i.e., the act by which a personal stamp is imprinted on a song, a personal note tending to add a "plus value" to the performed work, that of the performer's subjective interpretation.

Consequently, the act of performing a song amounts to topological transformations applied to the sound forms representing the song obtained by changing the parameters' values of the sound forms.

1.4.6. The musical instrument associated to human voice

In order to increase the power of a song's emotional modulation, musical instruments have often been called upon. In ancient texts and frescos, the lyre is usually associated with the song, to the "emotional modulated" speech. By adding an instrument to the human voice, the amount of "expressive information" increases with respect to the "conveyed rational information." Consequently, the amount of information conveyed through "resonance" increases. The melody and its "accompaniment" become more and more important relatively to the rational information contained in the respective text.

autonomy of the emotional modulation

The ever more frequent use of musical instruments associated with the human voice and their continual improvement, gradually leads to the moment when the instrument's sounds become the only bearers of the information conveyed by the sound language. By instrument-generated sounds, the psycho-emotional information "detaches itself" from the rational information and becomes autonomous and solely carried by the sound parametric aspect; we are witnessing the occurrence of the pure language of music. This language becomes "modality of expression," a language free of any logic-rational information to be decoded and understood. Hence, the musical instrumental language is a language exclusively acting through resonance. Unlike the song, the sound forms in musical instrumental language no longer convey concepts, no longer include words and phrases of rational meaning, but consist only of sounds of different duration, pitches, intensities and timbres. *By completely losing the logic-rational information, the musical language becomes the language with the highest degree of ambiguity and polysemy.*

With good reason, in a hierarchy dictated by the degree of ambiguity, a hierarchy established by Solomon Marcus (1.3.2), the instrumental musical language ranks first, followed by poetical language, then the common language and the scientific one.

1.4.8. Sets of timbres, intensities values and positions in the usual Euclidean space. The sound space with the deepest structure: the musical instrumental language space

The notation of sounds in European music has a finite scale of intensity values also, as a result of the two fundamental once: "piano" and "forte" (**p, f.**).

These intensity degrees certainly do not represent absolute values, but only the notation of the *loudness sensation* of the sound. By considering a sound of moderate loudness (*ML*) as the origin (*O*) of an intensity axis and denoting by –1 and +1 the piano (**p**) and forte (**f**) values, we will be able to write down the usual dynamic markings in musical scores by digital values, as illustrated in Fig. 1.9:

ppp	pp	p	mp	ML	mf	f	ff	fff
–3	–2	–1	–1/2	0	1/2	1	2	3

Fig. 1.9

ppp – pianississimo; **pp** – pianissimo; **mp** – mezzopiano; **mf** – mezzoforte: **f** – forte; **ff** – fortissimo; **fff** – fortississimo

On the whole, these values form the finite sets of intensities values with which the composer works, values which could be represented on a third coordinate of a three-dimensional sound space (1.1.4).

Implication of musical instruments in the process of generating "*musical sound forms*"results in the limiting of the timbres possible in these sound forms to those of the respective instruments. As a consequence, the set *D* of the timbre parameter (1.1.3) values characterizing a sound becomes, in instrumental music, a finite set. The "container" of timbres consists of the timbres of the musical instruments, each class of instruments having a fundamental specific timbre.[8] The specific of the set *D* elements obviously depends on the epoch and geographic space of the music under consideration: for instance, with baroque music, the organ, harpsichord, viola da gamba timbres are characteristic; for classical European music, the instruments of the symphonic orchestra; for Indian and Chinese music: the timbres of their specific instruments, etc.

Evidently new musical instruments, like the electronic ones, might increase the number of the elements of the set of timbres, the set remaining however finite.

Musical notation also allows the specification of the "mode" of "attack" of the sound sources. For each class of instruments, there is a relatively restricted number of attack modes. For instance, for string instruments, these are, as a rule, limited to those indicated by the expressions: "arco" (the string vibrates by means of the bow), "pizzicato" (the string is plucked with the finger), "col legno" (the string is struck with the wooden part of the bow), etc. The number of signs indicating the attack mode on different instruments has remarkably increased in contemporary music scores; however, in practice we should consider that in music, the set *E* of the modes of attack is finite.

[8] Here we do not take into consideration the fact that, in a class of instruments, for example the "violin class," each instrument individually may have certain timbre features characterizing it.

Theoretically, a set *F* of the stereophonic parameter values, the triplet (*X, Y, Z*) representing the possible coordinate values of the Euclidean space, are of the power of continuum. But, in practice, in a given concert hall, the possibilities of arranging the instruments are limited, especially for performing musical pieces of classical repertoire. Therefore, the set *F* for a given hall is in fact finite also.

1.4.9. The musical instrumental language: a Cartesian product of finite sets representing values of sound parameters

We remark that in traditional European instrumental musical language all sets representing possible values of sound parameters defining the sound forms of this language are, as a matter of fact, finite sets. Hence, we may state:

(1) Musical language can be defined as a Cartesian product of finite sets representing values of sound parameters.

(2) The sound space of the instrumental musical language has the deepest and most rigorously defined structure.

1.4.10. An attempt to form a hierarchy of sound languages depending on the space structure of the sound forms which represent these languages

The finite character or that of the power of continuum of the sets @, *A, B, C, D, E* (1.4.4) allows us to define a hierarchy of sound languages considering the sound space structure of the respective language field..

Mechanically – or electronically – generated sound languages (sirens, etc.); Pure emotional languages (groan, scream, etc.)	*A, B, C, D, E*, (@ = ∅) of the power of continuum
Natural spoken language	@ finite *A, B, C, D, E* of the power of continuum
Poetic language (utilizing meters)[9]	@, *B* finite *A, C, D, E* of the power of continuum

[9] Meters in the sense of duration structure: iambs, trochee, anapest, etc.

Vocal musical language (song)	@, *A, B* finite *C, D, E* of the power of continuum
Instrumental musical language	(@ = ∅) *A, B, C, D, E* finite

1.4.11. A possible hierarchy of natural spoken languages in terms of the value of the ratio concerning the amount of information conveyed through the combinatorial aspect and the amount of information conveyed through the parametric aspect of the word

Let *S* be a spoken sequence in a natural language whose terms are strictly denotative, also let us consider two reference languages: the strictly rational language and the strictly psycho-emotional language. We will try to establish the degree of belonging to the above mentioned reference languages of a sequence *S'* by considering the values of the ratio between the amount of information conveyed through combinatorial aspects (*Ca*) and that conveyed through parametric aspects (*Pa*) (1.3.4).

The remark that logic-rational information may be conveyed through (*Ca*) and may be denoted by the graphical symbols of the respective language, and that psycho-emotional information ("lost" in writing) may be conveyed through (*Pa*), allows us to make the following preliminary statement:

In a sentence, the greater the amount of information capable of being recorded with graphical symbols, the higher should be considered its degree of belonging to the strictly rational language; the greater the amount of information which may be lost through writing, the higher should be considered its degree of belonging to the strictly psycho-emotional language.

In order to define its degree of belonging to either of the two reference languages, we will have to find a mode of measuring the amounts of information (*a, Pa*) in the sequence *S*.

Let us suppose that, on the one hand, the quality of the psycho-emotional state of a person pronouncing the sentence *S* is known and that on the other hand, by analyzing the spectrum of the uttered *S* sentence, its sound parameters are defined.

In other words a correspondence between the psychic condition of a person pronouncing the sentence *S* (state of joy, anxiety, boredom, etc.) and the *mode* of expressing *S* (words pronounced loudly, whisper, slow and monotonous speech, etc.) i.e., the parametric values defining the *S* sound form, is established.

Let us also suppose that a number of quantitative measurements of psycho-emotional information conveyed through spoken languages, for instance, the deviations of the values of their sound parameters from the sound parameters values which define common speech, are introduced.

Let *IL* be the information *I* conveyed through the sound form representing a spoken sequence of a language *L*; let *IC* be the information conveyed through (*Ca*), and *IP* the information conveyed through (*Pa*).

By definition $IL = IC + IP$.

Consequently, if $IC = 0$, the information is conveyed exclusively through parametric aspects, the language is strictly psycho-emotional ($IL = IP$).

If $IP = 0$, the information is exclusively conveyed through combinatorial aspects, the language here being strictly rational ($IL = IC$).

Since the sum $IC + IP$ is constant, we can imagine different modes of representing the ratio between *IC* and *IP*. For instance, considering $IL = 1$, the value *IC* and *IP* are variable between 0 and 1, respectively. When considering the unit of measurement to be 0.01, we can then express in percents the "belongings" of a given sequence to one of the considered reference languages.

1.4.12. Two types of topological transformations of sound forms in the European musical language

In the previous sections we have studied the topological transformation of the musical sound forms viewed as a result of the act of performance. We tried to prove that the artistic interpretation of a musical fragment amounts to a topological mapping of the sound form representing the considered musical fragment.The domain of these mappings is relatively restricted and concerns exclusively changes of the sounds duration, pitch, intensity and timbre. What holds for musical fragment holds for a sonata, a symphony or the music of an opera.

Apart from transformations in performance, a category of much more complex topological transformations can be discovered in the act of musical composition, i.e., in the act of creating major musical forms: polyphonic pieces, fugues, sonatas, etc. These transformations find their

origin and support mostly in the graphical representation of the sound forms, in musical notation, as we will try to prove in the following chapter.

1.5. THE ROLE OF SEMIOGRAPHY IN THE DEVELOPMENT OF EUROPEAN MUSIC

1.5.1. Creative aspect of the written sign: graphical thinking

Instrumental musical language, defined by sound parameter values only, needed the invention of its own system for notation.

At first, by giving a second function to some letters of the alphabet, the ancient Greeks obtained a mode of writing down sound pitch: a letter placed over a written word, indicated the pitch in which the respective word (or group of words following it) had to be uttered. In this way, the mode in which a certain text had to be sung, "interpreted" as far as the intonation, i.e., the pitches of de various syllables were concerned, was settled. Gradually, musical notation progressed, reaching in the notation of Western European music its most advanced form.

Western European music is, no doubt, one of the most interesting phenomena of universal culture.

Among all musical languages, it is the one which became to the highest degree an independent art, an artistic reality in itself, an elaborate emotional language which surpasses the spontaneous folklore form, as well as that of art associated with feasts, rituals, theater or poetry.

Western European music generated an organized polyphony as a result of musical thinking; it is a musical language which, while surpassing the elementary problems of structuring the sound material, draws up sound forms of great complexity, works of art representing the thinking of a creator.

It is well known that speech is not only a means of communication, but a means of forming ideas also.[10] As Pierre Souyris states, "language far from being an instrument of our thinking, is its own substance and essence."[11]

Along with speech, writing is a powerful tool for thinking. "Thinking could never have been developed without the space support of writing..." "Thinking materialization is a secondary result of writing. Its main mission

[10] Henry Wald, *Realitate şi limbaj*, Ed. Academiei, 1968, p. 9.
[11] Pierre Souyris, *Langue et réification*, in «Esprit», 1967, p. 336.

has been to increase the generalization and abstraction rhythm." ... "Together with speech, writing is the main instrument by which abstractions are built", concludes Henry Wald.[12]

Considering musical thinking from this point of view, Western European music can be viewed to a great extent as the result of its mode of writing, of the graphical representation of sound material, giving birth and improving polyphony, harmony, serial music and the theory of musical forms. All these are undeniable proofs that musical thinking, speech and writing are firmly linked and mutually influenced.

Elaboration of musical language, as of any other higher intellectual activity, cannot be conceived without involving the writing mechanism.

One can easily see that musical language has improved as a result of its notation system, i.e., musical semiography. As one knows, the system was, at the beginning, so unprecise that it rather represented a way of remembering a melody the performer already knew.

By improving the means of writing down the pitches of the sounds (stave and "neumes")[13] as well as their time lengths ("measured music" – a system where the time values are obtained by dividing a fundamental unit), the notation of music has had a substantial evolution.

Finally, the idea of "tempo" led to the possibility of diversifying the displaying speed of a musical piece by modifying the time unit value.

1.5.2. The double function of musical semiography: means of notation and creation tool

Along with the development of musical notation, the stave, initially created to be the means of notation of conveying by writing musical structures, gradually became a means for sound organization, i.e., for musical creation.

Musical symbols, apart from their notational function, also acquire the function of structuring tools.

Polyphony could be obtained only on a notation system which makes possible the control of the simultaneous evolution of two or more "voices". Ancient Greek, Byzantine, and old Oriental music, not having a sufficiently developed writing mode, have remained in general in the linear notation phase and so could not create complex polyphonic music.

[12] Henri Wald, *op. cit.*
[13] Musical notation signs used during the Middle Ages.

In Western music, sound notation has generated musical thinking relying on the possibility of the organization offered by the stave and the symbols of the sounds which initially represented only graphical means of conveying emotional language.

Thus, the act of musical creation went beyond the phase of imagining some simple melodies written down only for the purpose of conveying them. Music started to become a higher level of sound organization due to a sophisticated logical thinking process.

1.5.3. Abstraction of the act of musical creation: the composer works with the image of the sound

It is an important fact that in the Middle Ages European composers started to think in an abstract way about the building of a sound world through symbols representing sounds whereas before this period the act of creating music had been a spontaneous one where emotion was directly materialized in song or instrumental playing. *The composer reached the thinking stage where he no longer worked only with sounds, but with their images represented on the staves.*

A first group of sounds, a first "musical idea," a first musical structure is "written," becoming an image on which one works, to which one applies different processes of transforming and linking up with images of other musical structures.

The composer creates the sound world through its space image.

The act of creation takes place on paper with graphic signs and symbols – with "*sound images.*"

The transition from sound to the sign defining it is similar to the change from concrete objects to the abstract concept of size and number which are the starting point of mathematical thinking.

1.5.4. The staff: ancestor of Cartesian coordinates. Centuries ago, composers had the intuition of multi-dimensional spaces

Committing the sound world to paper, the composer creates a restricted geometry where the relations between sound parameter values can be studied. The staff imagined by the musicians of the 11th century appears as a primitive bidimensional space. Of course, there are considerable differences between the musical stave and the coordinate system introduced

in mathematics by Descartes in the 17th century. Generally speaking, the staff defines a space whose dimensions are time and pitch. But it also enables the writing down of the dynamics and the modes of attack through symbols. Adding to these the fact that the various scores are intended for different instruments, each instrument having its own special tune (in orchestra scores, each stave system is the "carrier" of a certain instrumental sound), the sound representation on the staff allows for the simultaneous notation of the values of sound fundamental parameters: pitch, duration, intensity, timbre and mode of attack. *Hence, we may state that, centuries ago composers had the intuition of multi-dimensional space, i.e.,. of a space with five dimensions, and they very competently succeeded in writing down the values of sound parameters as a function of time, each parameter representing one of the five dimensions of the considered space.*

It is interesting to note that composers adapted this space to the specific requirements of the musical field. At first sight, the stave seems to be a bidimensional space in the usual Cartesian sense. But as a matter of fact, from the point of view of the pitch dimension, it represents a row of pitch values defined by the lines and the spaces between lines. In a "normal" state, the staff represents the sounds of the C major scale. Theoretically, it has eleven lines (Fig. 1.10).

Fig. 1.10

In practice, only parts of the eleven lines are used. Five lines are selected for the pitch characteristics of the sounds of various human voices or instruments; "clefs" mark the sounds C3 (middle C of the keyboard), F2 and G3. Pitches of the sounds on different lines and spaces are defined accordingly. The clef indicating certain pitches can be considered as an "origin" of the pitch's axis, corresponding to the origin of the ordinate's axis of the two dimensional Cartesian space (Fig. 1.11).

Fig. 1.11

The other sounds belonging to the chromatic scale are represented by "accidentals", such as #, b, x, bb, (sharp, flat, double sharp, double flat,

natural), by means of which the pitch represented by certain lines or spaces of the staff are transformed, i.e., the value of certain pitches is raised or lowered (Fig. 1.12).

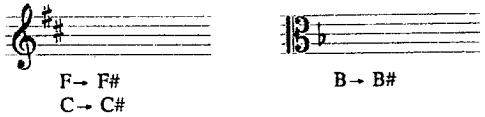


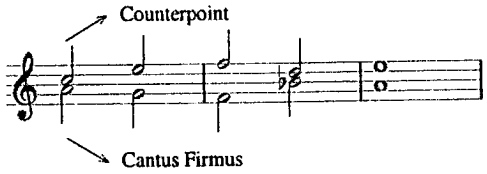
Fig 1.12

1.5.5. Two ancestors of the mathematical functions: "counterpoint" and "ciphered bass"

The first attempts to associate a given sequence of sounds, referred to as "cantus firmus", to another sequence of sounds, referred to as "counterpoint" are found in the 14th century.

Originally, counterpoint was limited to its simplest form, referred to in the polyphony treatises as "*note counter note*" (Fig. 1.13)

Fig. 1.13



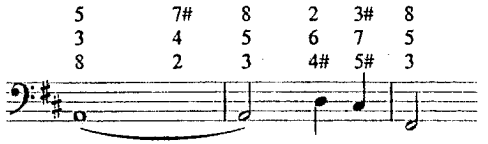
Soon, the operation extended to three, four and more voices. Developments of this mode of organizing a sound universe continued for several centuries and culminated with the monumental work of Johann Sebastian Bach, his *Kunst der Fuge*. Polyphonic writing becomes a science, requiring for the novice composer many years of training in order to perfectly master counterpoint. In mathematical terms, counterpoint (the word's etymology should be found in the expression "Punctus Contra Punctum")[14] can be expressed as: "the method of associating a point (representing a sound) to another point (representing another sound) according to a given rule. Rules have evolved down the centuries, the evolution being determined by the musical taste of that time and of the composer's genius. However, the idea of the involved mathematical notion is always there: *the counterpoint is a function which induces the correspondence between two sounds.*

[14] In this expression, the point should be understood as "note", the sound sign on the staff.

It is important to recall the fact that from the psychological point of view, music acts by the relationships between the sounds of a set, and not by the set of sounds itself.

In creating a melodic line, the composer actually creates a sequence of musical intervals and not a sequence of musical sounds.

The same situation occurs in the image of a chord, which is not conceived as a set of simultaneous sounds, but as a family of simultaneous pitch intervals, having a common origin. The undeniable proof of this mode of thinking is given by the many scores of J. S. Bach and of other contemporaries where the chords are marked by ciphers (Ciphered Bass), each cipher being the symbol of a musical interval. In this system of notation the ciphers 2, 3, 4, 5, 6, 7 are not values but symbols of the intervals of second, third, fourth, fifth, sixth and seventh, respectively (2.1.1) (Fig. 1.14).

Fig. 1.14

It is obvious that in the composition process called constructing a "Ciphered Bass" by which one associates to a sequence of sounds (bass) one or several other sequences of sounds (according to a law expressed through the symbols of musical intervals), the composer of Baroque implicitly worked with the concept of "*function*" from mathematical analysis.

As time goes on, attempts to incorporate emotion in sounds and to create an organized language increase. Gradually, as a certain ability is acquired in handling musical sound symbols, the act of creation is spent more and more "*at the composer's desk,*" in front of a sheet of paper with staves. By studying different possibilities to link and superimpose sound images, the composition becomes a science of associating sound symbols. The staff becomes the "laboratory" where the mind tries to give new forms to the sound material; the number of "composition rules" increases, the rules becoming aesthetical standards according to the taste of the different periods in which they appear.

The musical semiology, from a quite simple sound notation becomes a true crucible where the composer's imagination tries to obtain new gems.

1.5.6. The composer and the staff geometry. The concept of "transformation" appeared in music as early as the 13th century

A series of composition methods clearly indicate the transformational "work" done by the composer on the stave geometry. We will cite some of them:

1.5.6a. *"Alterations" of certain sounds of chords by raising or lowering them a semitone through sharps or flats, method known in the classic harmony treatises and defined as "transformations" by some modern theoreticians like Arnold Schoenberg*[15] (Fig. 1.15)

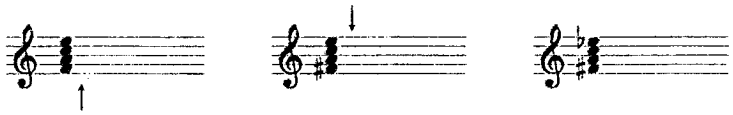


Fig. 1.15

1.5.6b. *Enharmony, i.e., the substitution of descendingly "altered" sound (1.5.4) with the inferior neighbor sound of the scale ascendingly altered or vice versa* (Fig. 1.16)

Fig. 1.16

The method of modulation referred to as "Enharmonic" allows a sharp but very spectacular transition from a given key to a very remote one; it is frequently used by the composers of the romantic epoch (Fig. 1.17).

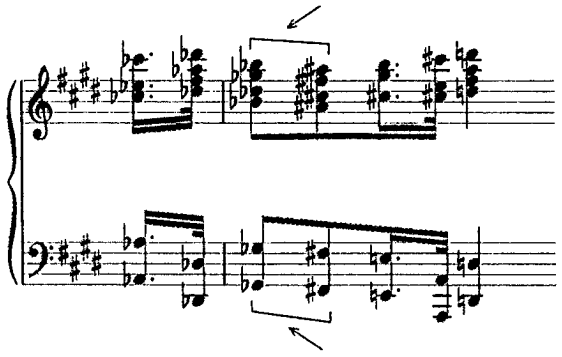


Fig. 1.17
J. Brahms – Fourth Symphony.

[15] A. Schoenberg, *Structural Function of Harmony*, W. W. Norton and Co., New York, 1969.

1.5.6c. *The interval and chord "reversion" through transposition to octave of the sound situated at the base of the respective interval or chord (direct condition, reverse I, reverse II, reverse III,* Fig. 1.18a,b)

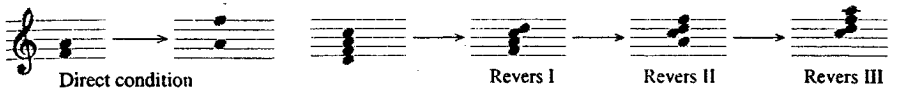


Fig. 1.18a

Fig. 1.18b

1.5.6d. *Proceedings with the sound image on staff: "reversions," "inversions" or "reversions of the inversion" of musical themes*

Some of these operations appear as far back as the 14th century in the works of Josquin des Prés, Guillaume de Machaut, Ockeghem or Villaert, and were magnificently developed by Johann Sebastian Bach in his "The Art of Fugue," and taken over and brought to a more advanced stage by A. Schoenberg's Serial School (Fig. 1.19).

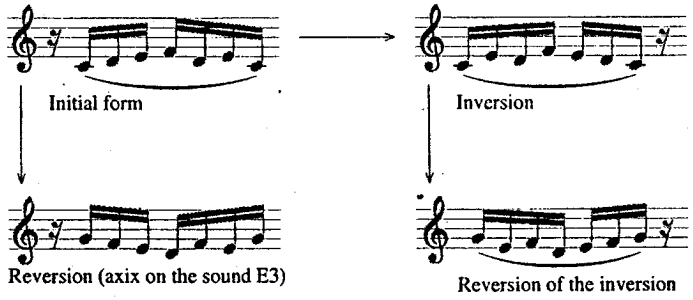


Fig. 1.19

1.5.7. Combinatorial aspects in structuring the musical time

As far as the time parameter is concerned, it is remarkable that in music the duration of the sounds and "rests" are recorded through symbols instead of intervals defined on a time axis. These symbols, associated with the "tempo" concept, form an extremely advantageous notation mode of musical time from the point view of the "paper" space. For instance, if the symbols were to be replaced on the time axis with intervals of values

proportional to the duration, the paper required for Beethoven's Ninth Symphony would probably fill a library.

The symbols for writing down the duration of sounds and rests form a spatial representation of time. The image of musical duration has allowed the drawing up of the various rhythmic formulae representing the support of the musical discourse. Knowing that rhythm is a structure of duration, the possible combinations of rhythms that can be formed out of the set S of 14 symbols (Fig. 1.20) representing the duration of sounds and rests, are surprisingly great.

Fig. 1.20

As known, if we wish to build out of a set of n elements a structure of k elements – any element being allowed to be repeated – the number of possibilities is nk. Consequently, a rhythm consisting of 3 duration (sounds or rests) can have 2,744 hypostases, and another one built of 6 duration may have 7,529,536!

In the above, we have only considered duration representing binary divisions of a fundamental unit (whole note). But traditional music notation also has triple divisions and those expressed through "quintuples," "sextuples", etc.

Contemporary music has added the so-called irrational divisions, for instance, 5 units unfolding on the interval defined by seven 1/16 duration symbols, Fig. 1.21.

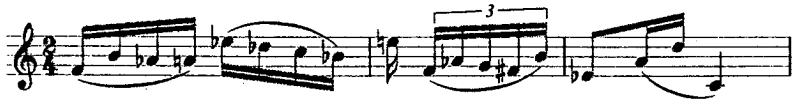



Fig. 1.21

In these conditions, the number of rhythms that can be generated through musical symbols is much greater!

Besides the combinatorial aspect of rhythms generation, there is also a transformational aspect, usual in the baroque polyphony, i.e., the "*increase*" or "*decrease*" of the duration of a rhythm, of a theme or of a musical phrase.

These transformations are obtained by making use of symbols representing duration of double or half-value, respectively, as against the initial rhythm, theme or phrase duration (Fig. 1.22).

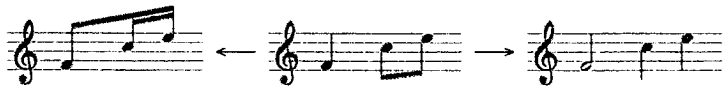



Fig. 1.22

Olivier Mesiaen has introduced a method of increasing the time length of a musical fragment by the so-called "valeur ajoutée" (added value), process increasing the value of a duration symbol by the associated value.

As a particular case, a point added to a note increases its duration by half of its initial value (Fig. 1.23).

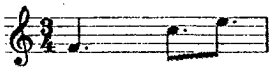
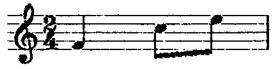


Fig. 1.23



1.5.8. Chronological time value of the symbols denoting duration

By "tempo" indications regarding the mode of unfolding of the "beats" measuring the duration (the interval between two consecutive beats representing the unit of measure), the composer transforms the value of duration symbols. At first, the frequency of these beats was defined by concepts specifying a certain speed (Lento, Allegro, Presto, etc., or their equivalents in other languages). By means of the Maelzel metronome, the time interval between two successive beats is defined with respect to the minute. The time length of a symbol S is indicated by the frequency of the beats, for instance, a $S = 124$, should be understood as $S = 60$ s/124.

The "tempo" transformation changes the musical discourse – from the time parameter point of view.

1.5.9. Conclusions

In the above we have tried to prove that:

1. On the whole, the composer does not work with the sound, but with its image. As time has been passing, he has been in search of ways for dealing with the sound images, i.e., with the symbols of the sounds. In this

respect, "there is no difference in the act of creation between the scientist and the artist," as A. Moles remarks.[16]

2. Traditional Western European musical language, as an act of thinking, represents a set of methods, transformations and linking of sound images carrying emotional potency.

3. The possibilities offered by Western European musical semiography have widely determined the sense in which this musical language has been developed. It is quite possible that another means of writing down musical sounds would have determined the development of Western European language in a different way than the one we know today. In this sense some linguists and logicians claim that had Aristotle spoken Chinese, the Aristotelian logic would have been different. More primitive notation couldn't have generated a musical thinking and would have remained a simple method of writing, of graphical setting up.

4. Traditional Western European music developed on the principles of some initial sound sets – sound forms referred to as themes, motifs, etc. – obtained through arrangements with repetition of some elements pertaining in turn to certain finite sets of sounds, for example, the sounds of a musical "*mode*". In the musical composing process, these sets undergo various specific transformations. In certain cases, the above-mentioned sets include only one element, as in the case of generating intervals and chords.

In the following, after summing up the elementary transformational music composition methods as well as their performance in the sound space R^2, we will try to build a general model for the investigation of these methods, a model allowing: (1) to define the character of these transformations, the classes they belong to and to isolate the particular cases used up to now in music; and (2) to approach the problem of generating sound universes wherein these classes of transformations could be exhaustively used in the process of musical creation.

[16] A. Moles, *La création scientifique*, Genève, 1957, p. 48.

2. MUSICAL CONCEPTS AND COMPOSITION METHODS INTERPRETED IN REAL SOUND SPACE R^2 AND IN VIRTUAL SOUND SPACE R^3

2.1. MUSICAL SOUNDS AND MODES IN SOUND SPACE R^2

2.1.1. Musical sounds represented in sound space R^2

We recall that sound space $\mathbf{R^2}$ has as dimensions time (t) and sound pitch (p) (1.5.5) Depending on the geometrical or pure musical context, we will use as symbols of the sound space $\mathbf{R^2}$ coordonates either x, y or t, p.

In order to express numerically the musical sound configurations in $\mathbf{R^2}$, we will introduce as the unit of measure on the time axis (Ox) the thirty-second note and on the pitch axis (Oy) the temperate scale semitone, resulting from division of the octave into 12 equal parts. By "octave" is understood the pitch interval between a sound S_1 of frequency F and a sound S_2 of double frequency ($2F$).

The time axis origin is an arbitrarily chosen moment. As we will see further on, due to some transformations applied to sound forms in $\mathbf{R^2}$, certain sounds might appear with abscissae having negative values, hence appearing on the "left side" of the origin.

The pitch axis origin is at the point corresponding to the sound C_3 pitch, the sound C in the center of a piano keyboard.

The musical sounds appear in the sound space as curves of $p = f(t)$ equations. The points A and D of abscissae a,d, (the origin and extremity of the curve representing the sound), define the appearance and disappearance moments of the respective sounds.

As a rule, these curves are colinear sets of points i.e., straight line segments. The sounds thus represented are "rectilinear" sounds: S_1, S_2, S_3 in Fig. 2.1.

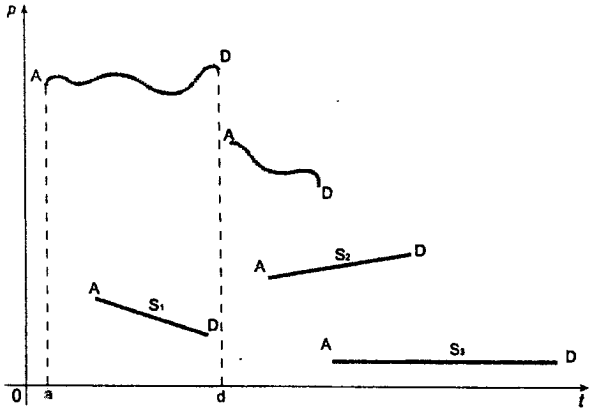


Fig. 2.1

2.1.2. Rectilinear sounds considered as vectors in sound space R^2. "Glissando" sounds. Clusters

Considering on the one hand that a straight line segment representing a sound is generated by the trace left in $\mathbf{R}^2$ by a sound point in motion,[1] and on the other hand that by definition, time is displayed on the time axis in a positive sense, hence the sound point in motion goes over the sound space from "left to right," a rectilinear sound may be represented in $\mathbf{R}^2$ by a vector (V) whose origin is in A and extremity in D.

The vector has as support the line defined by the points A and D. The length of segment AD is the vector's module.

The duration Δ of the sound represented by the vector AD is equal to the value of the interval defined by the vector projection on the time axis (Fig. 2.2).

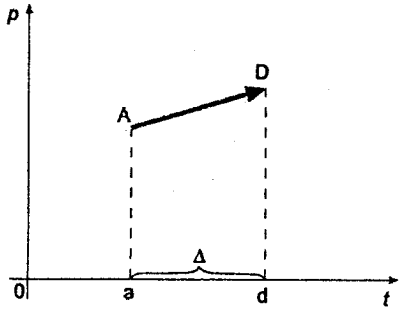


Fig. 2.2

[1] According to Jordan's theory, the graph of a curve in $\mathbf{R}^2$ can be considered as being plotted by a point in motion.

For the sake of simplicity, we will represent the rectilinear sound as straight line segments, and not as vectors, except in certain cases.

Usually, musical sounds have a constant pitch while they unfold. Consequently, the vector or straight line segment representing a musical sound is parallel to the time axis (S_3 in Fig. 2.1).

But, in music, the "glissando" sound is also used. By "glissando" is understood a sound whose pitch is continously transformed covering a pitch interval in a time interval. Depending on the direction the sound is displayed with respect to the pitch axis, the glissando sound is ascending or descending. Generally, in traditional music, the glissando links two sounds of different constant pitches (Fig. 2.3a, b).

The glissando "starts" somewhere during the first sound's duration. The moment of starting the glissando is, as a rule, left to the performer's decision. Contemporary music, sometimes uses the "individual" glissando sound in the sense of a sound changing its pitch without representing a link between two sounds of different constant pitches (Fig. 2.3c, d).

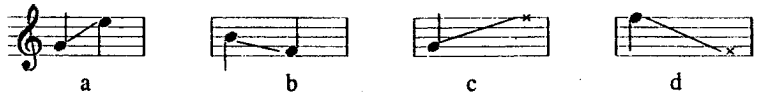


Fig. 2.3 a,b,c,d

In sound space $\mathbf{R}^2$ a glissando sound may be represented as a strictly increasing or strictly decreasing continuous function. We will refer to the glissando sound as "linear" when it can be represented in $\mathbf{R}^2$ by a linear function. We will call amplitutude (AM) the interval defined by the projection of a glissando sound on the pitch axis (Fig. 2.4).

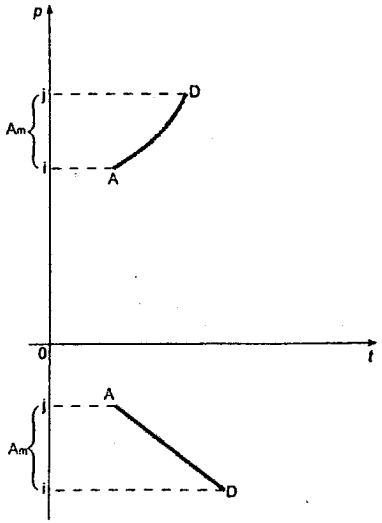


Fig. 2.4

In this work we will use, generally, linear glissando sounds.

Let us consider two consecutive sounds succeeding without interruption, the disappearance moment of the first coinciding with the appearence moment of the second (Fig. 2.5).

Fig. 2.5

Their correct representation in $\mathbf{R}^2$ should appear as closed intervals left side and open right side (Fig. 2.6).

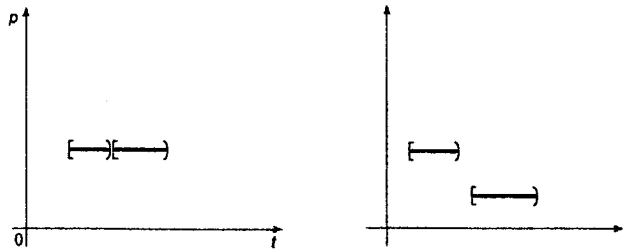


Fig. 2.6

For the sake of simplicity, we will represent the sounds as straight line segments only, except in the case of a sequence of sounds of the same pitch, as shown in Fig. 2.7.

Fig. 2.7

Several consecutive sounds of different pitches may be also represented as in Fig. 2.8, where the line segments perpendicular to the time axis represent null sounds of zero duration (1.1.3).

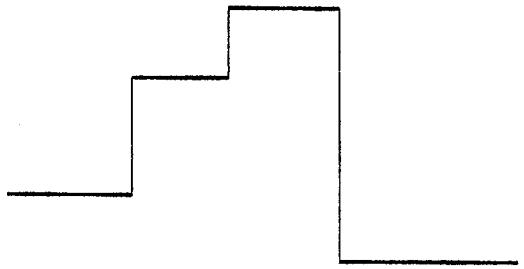


Fig. 2.8

In contemporary music, there also occurs the sound referred to as "cluster", characterized by the fact that, in a time interval, an interval of

pitches (which might be continuous or consisting of a number of consecutive semitones belonging to the temperate scale), is covered.

A cluster is defined by inequalities of the form

$$a \le t \le d$$

$$i \le p \le j$$

where i and j are the cluster points of maximum and minimum pitch.

When the cluster consists of a linear sound, it appears as a null sound with zero duration (1.1.3).

Sounds of different types may be concatenated.

Fig. 2.9 illustrates various types of sounds represented in the sound space $\mathbf{R}^2$:

(a) nonlinear sounds represented by curves;
(b) linear sounds of constant pitch;
(c) nonlinear "glissando" sounds;
(d) linear "glissando" sounds;
(e) "cluster" sounds;
(f) concatenated sounds of different types.

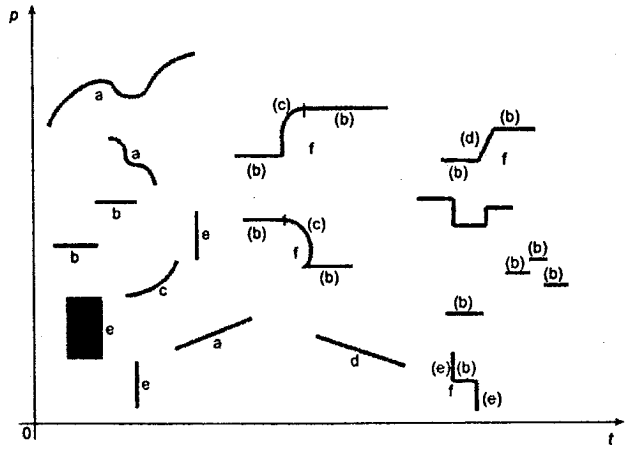


Fig. 2.9

Let S be a linear sound. We may define S in $\mathbf{R}^2$ by the coordinates of points A and D (2.1.1) which will be written as $S(xA,\ yA,\ xD,\ yD)$, where the expression $xA,\ yA,\ xD,\ yD$ represents the DV definition vector of S. We will write DV in the simplified form $x\ y\ x\ y$.

In Fig. 2.10 the sound S_1 is defined by the vector

3 5 10 5

Since $pA = pD$, the sound has a constant pitch.

The sound S_2 is defined by the vector

7 –4 13 2

Since $pA \neq pD$, S_2 is a glissando sound.

The sound S_3 defined by the vector

–7 [2 – 9] –5 [2 – 9]

is a cluster of duration

$$-7 - (-5) = |2|$$

expressed by a sixteenth note.

The sound S_4 defined by

–4 (–9 – (–2)) –4 (–9 – (–2))

is a cluster of zero duration (null sound) (1.1.3), since $xA = xD$. For this type of sound we introduce the notation γ.

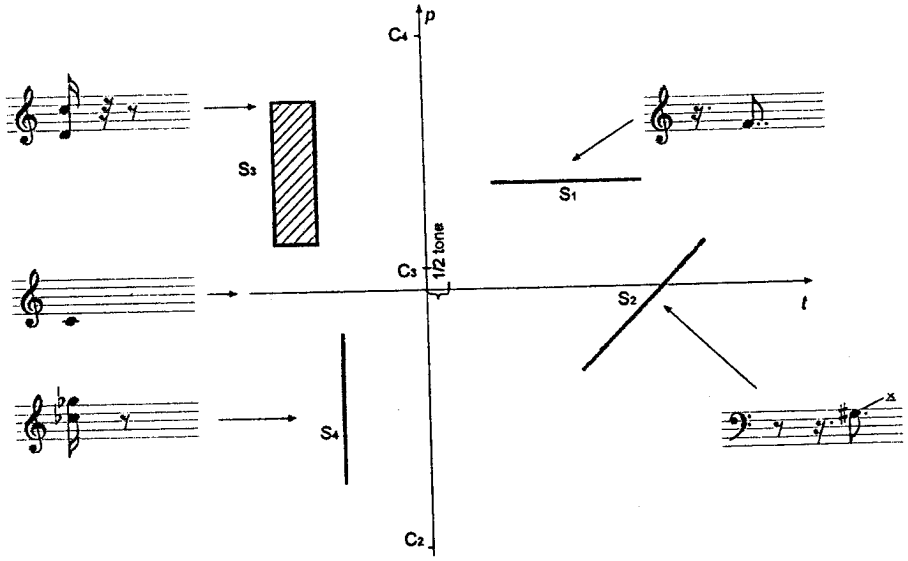


Fig. 2.10

2.1.3. Representation of musical sound forms by matrices

According to definitions (1.1.3), a musical sound may be viewed as a sound form having only one element, defined by a vector (2.1.2). A sound form is composed of a set of n elements and is defined by a matrix of four columns and n lines, each line representing a sound definition vector DV.

Thus, the sound form illustrated in Fig. 2.11a is numerically defined by the matrix in Fig. 2.11b and represented in sound space $\mathbf{R}^2$ by Fig. 2.11c.

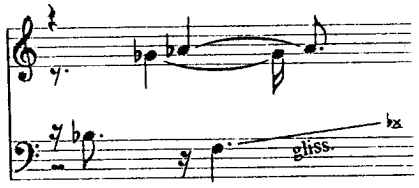



Fig. 2.11a

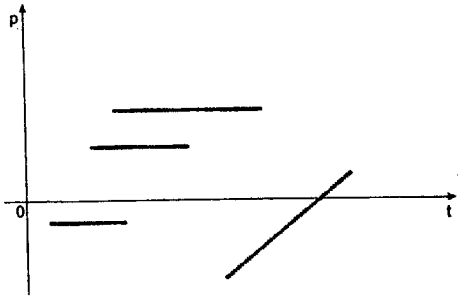


Fig. 2.11c

x	y	x	y
2	-2	9	-2
6	6	16	6
8	8	22	8
18	-7	30	2

Fig. 2.11b

2.1.4. The value of a pitch interval may be expressed by the absolute value of the difference between two sound point ordinates

In acoustics a pitch interval between two sounds was at first defined by the ratio of the strings lengths producing the respective sounds (Pythagoras), and, subsequently by the ratio of the number of oscillations generated in a second by the sound sources (Helmholz).

In music, a pitch interval is defined by the distance between the steps of the temperated scale: one and the same step forms the unison; between two neighbouring steps an interval of second; between three consecutive steps, an interval of third, etc. Within a temperate scale, the distances are measured in semitones (2.1.1). The musical theory shows that the minor second is made up of one semitone, the major second of two semitones, the minor third of three, etc. The musical interval may occur between two simultaneous sounds or separated by a time interval (harmonic and melodic intervals). The intervals within an octave are considered to be simple

intervals. The *composed* intervals, ninth, tenth, eleventh, etc., are obtained from a simple interval added to the octave.

Within the octave, a musical interval may be expressed by the absolute value of the difference between two sound point ordinates. If the interval exceeds the octave, its value is defined by the remainder resulting from the division of the ordinate difference by 12 (12 being the octave number of semitones).

For example let P_1 and P_2 be two sound points of ordinate 9 (pitch A_3) and 20 (G_4#), respectively. The difference of the ordinates is 11, the number which defines the major seventh pitch interval. Let P_3 be a sound point of ordonate –7 (F_2). The ordinates difference of P_1 and P_3 is 16 semitones. Dividing this value by 12, the remainder defines the major third interval.[2]

The same rule applies to two sound points having negative ordinates. For instance $p = -2$ and p= –9 will represent the interval of fifth E_{b2} B_{b2}.

Similarly, the duration of a sound in the sound space $\mathbf{R^2}$ is defined by the absolute value of the difference between the abscissae of the points A and D defining the respective sound (Fig. 2.12). For instance, the points P_1 and P_2 having the abscissae –7 and 9, respectively, define a sound having a duration expressed by a half note (16 thirty-second notes (2.1.1.)).

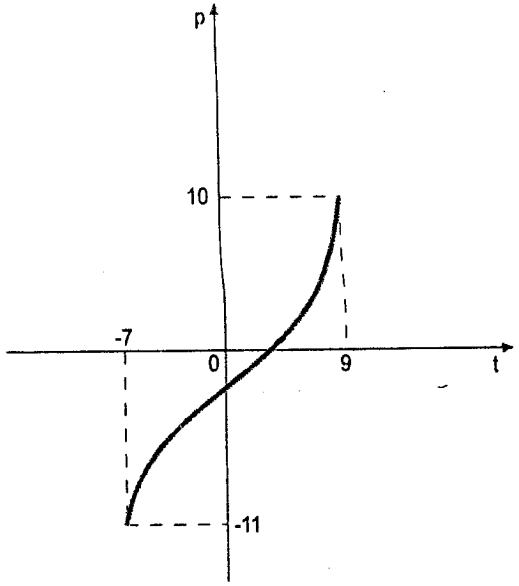


Fig. 2.12

[2] The intervals of 13 and 14 semitones can be considered also as minor and major ninths, respectively; those of 15 and 16 semitones as minor and major tenths.

2.2. TOPOLOGY OF MODES IN REAL SOUNI SPACE R^2

2.2.1. Continuous structure of sound space R^2 Discontinuous structure of musical sound space A short journey into the world of modes

In the sense of the concept of continuity defined under (A.1.1; A.1.2 the sound space continuity means that any point is, or may become, th support of a non-null sound event. As the space $\mathbf{R^2}$ is a continuous space the set of sounds of different pitches that may be represented in this space i of the power of continuum. In other words, the composer may write dow sounds of any pitch.

Throughout history, music of different cultures was conceived b making use of some finite sets of sounds of different pitches, the so-calle "modes."

From the first modal systems mentioned by Plutarch, the "oligochordies of 1, 2, 3 and 4 sounds, passing through the penta-, hexa- and heptachordi ancient Greek, Gregorian and Byzantine modes to the major or minor key or to those representing parts of the set of 12 sounds of the temperat chromatic scale, folk music and composers have made use of "modes" an to a great extent continue to do so.

The mode structure is defined by:

1. the number of sounds of the considered mode;
2. the value of the pitch intervals between two consecutive sounds c the mode, when these are arranged on the pitch scale in a strictly increasin sequence (sometimes strictly decreasing as in Byzantine music).

2.2.2. Different units of measurement for intervals separatin the modes' consecutives steps

It is important to emphasise the fact that, except for the modes whic represent parts of the temperate chromatic scale, where the intervals betwee two consecutive steps are measured by the "temperate semitone," in mo other modes these intervals are measured with different units. An example given by the major and minor keys themselves, particular cases of mode where the diatonic semitone value is different from that of the chromat semitone (2/5, 3/5 of a tone, respectively).

In ancient modes, it is difficult to establish exactly the value of intervals between the sounds constituting them. Also, it is very possible that in many cases, for a given mode, the interval value between steps vary from one instrument to another, on account of the construction of the respective pipe or lyre.

Heedful studies have been made of Byzantine music modes, as well as of various folk musics; most of them prove that the interval value between the steps of the respective modes are different from the temperate semitone of the European classical music. Two examples may be given:

1. for Byzantine music, the unit of measurement of the intervals between steps results from an octave divided into 68 micro-intervals. The intervals have, respectively, the values of 9, 7, 12, 12, 9, 7, 12 micro-intervals;

2. for old Romanian music, Anton Pann has simplified this scale by reducing it to 22 micro-intervals, by introducing the concepts of "large tone" (4 divisions), "small tone" (3 divisions) and "smaller tone" (2 divisions).

As a rule, folk music uses modes of 7 sounds at the most. But, in the music of some ethnic groups, one may find modes with more than 7 sounds and even over 12, the number of sounds of the dodecaphonic mode. For example, Arabian traditional music has a mode made up of 16 sounds, defining intervals of various sizes covering the octave interval.

Finally, through electronically generated sounds, one can imagine modes based on intervals obtained by dividing the octave by divisors other than 12, by dividing other intervals than the octave (for instance the tenth) by various divisors or by arbitrarily defining the value of each interval between two consecutive sounds of the mode.

2.2.3. The purpose of the journey in the world of modes revealed. Musical modes viewed as topologies in the sound space $\mathbf{R}^2$

We have travelled across the world of modes in order to show that the sound space of traditional music, which makes use of the concepts of mode and keys, is a *discontinuous space*. When a musical structure conceived in a certain mode is represented in the sound space $\mathbf{R}^2$, by definition, the sounds of the respective structure will have as ordinates only those values which correspond to the sound pitches of the considered mode. By resorting to the concepts of null and non-null sounds (1.1.3), we will say that the

representation of the modes in $\mathbf{R}^2$ induces in this space a *modal structure* characterised by "*null sound zones*" and "*non-null sound zones.*" The null sound zones may be considered as zones wherein, by definition, the pitch parameter (frequency of sound waves) has zero value. The non-null sound zones correspond to the sounds of the respective mode.

An example intuitively illustrating this fact is the representation in $\mathbf{R}^2$ of the sound space of the temperate chromatic scale. Its structure could be imagined as being equivalent to the structure of the sounds generated by the piano keyboard (non-null sound zones). Between two adjacent piano keys, one can, theoretically, insert an infinity of keys generating sounds of frequencies included between the sounds of the two keys (null sound zones).

The null and non-null zones, parts of the sound space, form a topology in R^2.

The non-null sound zones are open sets. They occur as sets of rectilinear points parallel to the time axis and represent boundaries between two sets of points belonging to null zones. The sets of null sound points, complementary to the union of sets of non-null sound points are, by definition, closed sets. In this sense, the musical modes may be considered as topologies in the sound space $\mathbf{R}^2$.

Inducing a modal structure in $\mathbf{R}^2$ amounts to the operation of defining the sets of points making up the non-null and null sound zones. Fig. 2.13 illustrates a sound space where a topology characteristic of a mode of 6 sounds is induced.

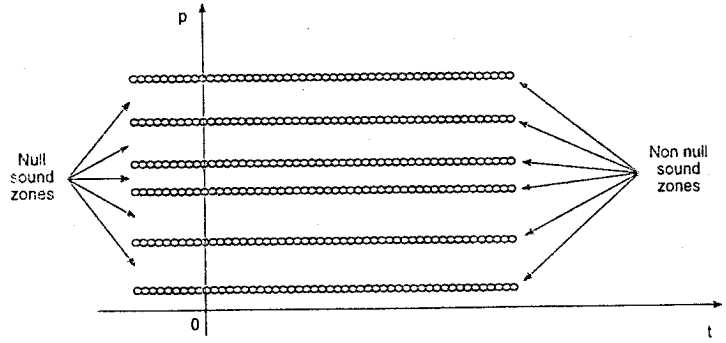


Fig. 2.13

Conferring to the sound space $\mathbf{R}^2$ the structure of the *C* major key amounts to a mapping that induces in this space non-null sound zones corresponding to the sounds of the white keys of the piano, of ordinates equal to those of the ordinates of the *C* major scale sounds (Fig. 2.14).

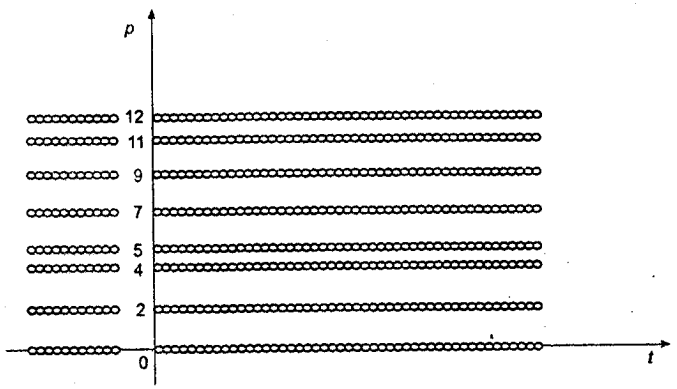


Fig. 2.14

2.2.4. The set of modes with a number *n* of sounds is of the power of continuum. Related to the respective number the set forms a class of equivalence

It is easy to remark that, with respect to the group of topological transformations, all the sound spaces wherein various modal structures of n sounds are induced are topologically equivalent (Fig. 2.15).

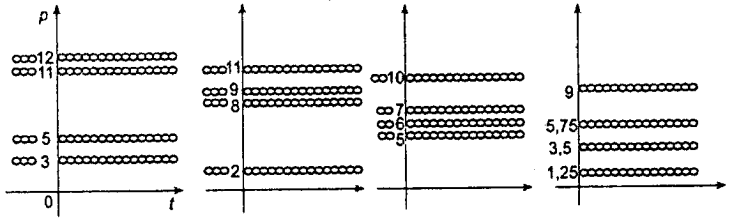


Fig. 2.15

Since in the sound space $\mathbf{R}^2$ the values that potentially non-null sound zones may take on the ordinate axis are of the power of continuum, the set of topological spaces with modal structures of n sounds is also of the power of continuum. With respect to the number n, the sounds constitute classes of equivalence. Consequently, the modes of n sounds in old, traditional or contemporary music are elements of these classes. *The set of modes of n sounds to be represented in the sound space* $\mathbf{R}^2$ *is of the power of continuum.*

2.2.5. Mode representation through vectors

Usually, when dealing with the sounds of the temperated scale, mod are represented by notes on the stave, or by digital symbols, each symb being associated to a sound on this scale. These representations are qui adequate as long as the modes are considered subsets of the set of 12 soun of the temperate scale. Allowing for the fact that the ordinates defining th non-null sound zones in $\mathbf{R}^2$ space may take continuous values on the pitc axis, it appears more advantageous to define modes by vectors, the vect components representing the ordinates of the considered mode sounds.

Let M_n be the set of topological spaces of modal structure of n sound An element of the set M_n, hence a mode of n sounds, will be defined by vector with n components, each component representing the ordinate of or of the mode sounds. For the best representation of the modes in the spa $\mathbf{R}^2$, especially of the modes not made of intervals obtained by dividing th octave into equal parts, it is preferable that the sound ordinates should b expressed in Hz, i.e., that the sounds should be defined by the frequency the sound source oscillations generating them.

Here are two examples: a mode formed of equal intervals (a) and mode made up of sounds whose pitches are arbitrarily chosen (b).

a. Let us consider the Javanese mode Slencho, whose intervals a defined by dividing the octave into five equal parts. Considering the mo unfolds on an octave defined by sounds of F and $2F$ frequencies (2.1.1), th vector V_1 defining the respective modal structure in $\mathbf{R}^2$ will have the for $V_1 = F * (2^0,\ 2^{1/5},\ 2^{2/5},\ 2^{3/5},\ 2^{4/5},\ 2^{5/5})$.

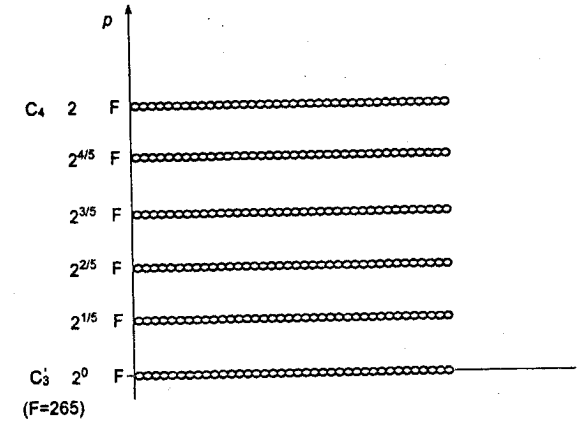


Fig. 2.16

Fig. 2.16 shows sound space $\mathbf{R}^2$ endowed with a modal structure defined by the vector V_1, where 265 represents the number of oscillations per second corresponding to the sound C_3.

b. The vector V_2 defines the modal space of a micro-tonal mode of 11 sounds, with intervals defined "step by step". The digital symbols represent values in Hz; the letters below represent the sounds of the temperated scale corresponding to the respective frequecies.

$V_2 = 285;\ 297.45;\ 320;\ 330;\ 342;\ 353.73;\ 360;\ 382;\ 415;\ 445.67;\ 459$

$\uparrow$ D $\quad\quad$ $\uparrow$ F $\quad\quad$ $\uparrow$ A

Fig. 2.17 shows the sound space $\mathbf{R}^2$ with modal structure defined by the considered vector V_2.

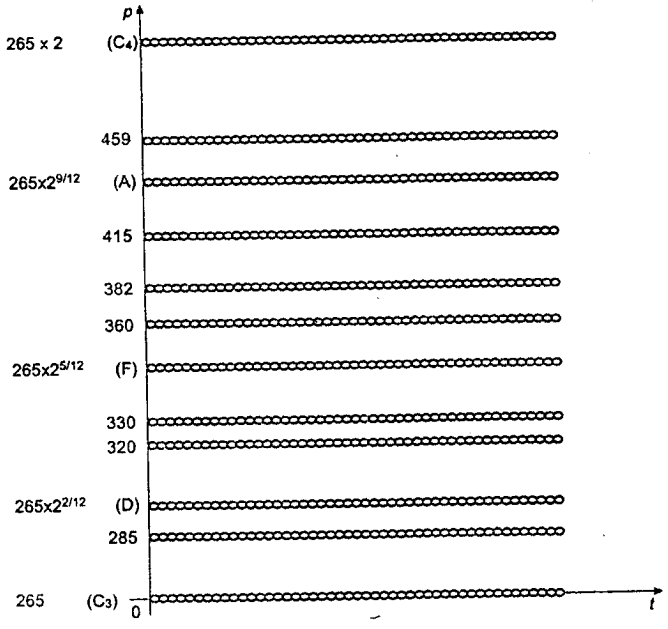


Fig. 2.17

2.2.6. Difficulty of including in discrete spaces and by combinatorial methods the various modal categories in a general unitary system

Studies about musical modes have in general been made by considering values categories, for instance, ancient modes, Gregorian and Byzantine ones, the major and minor keys, those obtained through the temperate chromatic scale, or through sequences of arbitrarily defined intervals, etc. In most cases, the studies have had an "a posteriori" character, aiming only at cataloguing and arranging the different types, without trying to integrate all of them in a general unitary system.

The difficulty in initiating such an approach mainly arises from the fact that the modes are considered as finite sets of sounds, their various structures being regarded as a combinatorial problem. In addition, since the modes are structures of intervals and not of sounds[3] and the units of measurement defining the values of these intervals vary not only from one category to another, but even within the same category, it is obvious then that certain operations with elements of different sizes become impossible.

2.2.7. Towards a possible general theory of musical modes by the instrument of continuous space topology

The representation of musical modes in sound space $\mathbf{R}^2$ and the surveying of their behaviour under the action of the group of topological transformations allow the integration of the various modal categories into a general system.

Since continuous values may be defined on an ordinate axis, the different units for measuring the intervals between sounds in the various categories, may be homogenized by expressing the sound ordinates in Hz.

Let M_5 be the set of all the topological spaces having a modal structure corresponding to the modes of 5 sounds. The number of M_5 being of the power of continuum (2.2.4), it includes all spaces endowed with modal structures representing the ancient Greek and Byzantine pentacordic modes, the pentatonics scales of different ethnic groups (for instance Romanian or Chinese), as well as all the modes of 5 sounds that may be built from the set of the temperate chromatic scale.

2.2.8. A set of still unexplored modes may become "building material" for the composer

The previous reasoning may be extended to any set M_n representing the class of equivalence of topological spaces relevant for modes of n sounds.

[3] For example, in the theory of music, the major mode is not defined by the sounds C, D, E, etc. or their transpositions, but by the structure of intervals characterizing it: tone, tone, 1/2 tone, tone, tone, tone, 1/2 tone.

Since only some numbers of the elements belonging to M_n are isomorphic with the modes of n sounds listed by the music theory treatises, one concludes that the difference between M_n and the subset of modes of n sounds already known represents practically an infinite number of new modes that can help the composer in his creative act.

In the following chapters, we will try to prove that certain methods of musical composition can be interpreted as topological transformations applied to musical sound forms in the real sound space $\mathbf{R}^2$ and in the virtual sound space $\mathbf{R}^3$.

The applied methodology aims at grasping the isomorphism existing between certain transformational methods some composers work with on the stave geometry and certain transformations specific to topological spaces with vectorial, projective, affine, Euclidean and metric structures. The order in which these transformations are presented is dictated by musical reasons, i.e., by a certain hierarchy of the musical structures under consideration, and not by a hierarchy induced by the geometrical structure of the space.

2.3. MUSICAL TRANSFORMATIONS IN SOUND SPACE R^2 ENDOWED WITH TOPOLOGICAL, VECTORIAL AND PROJECTIVE STRUCTURES

2.3.1. Mapping of a sound of a certain image into another sound of a different image amounts to a topological transformation of a curve in R^2

In real sound space $\mathbf{R}^2$, a sound is represented as a curve. Consequently, the mapping of a sound of a certatin image into another sound of a different image, amounts to a topological transformation of a curve in $\mathbf{R}^2$ (A.1.1) (1.2.2) (Fig. 2.18).

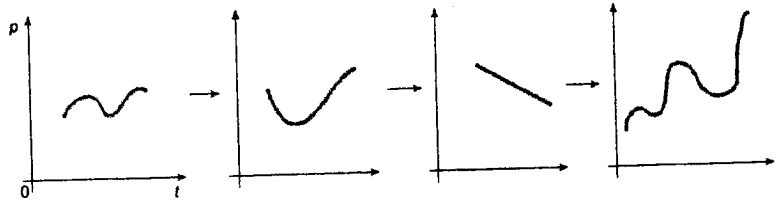
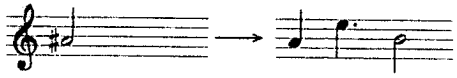


Fig. 2.18

Remember that, if there is a topological correspondence between elements of a set of curves representing sounds in the space $\mathbf{R}^2$, then t respective set forms a class of equivalence (A.1.1). In this context, t sounds in Fig. 2.18 are equivalent. Although it might seem surprising for t composer, a transformation of the type shown in Fig. 2.19

Fig. 2.19



may be interpreted as a topological transformation, if both images a topologically equivalent (1.2.2), Fig. 2.20.

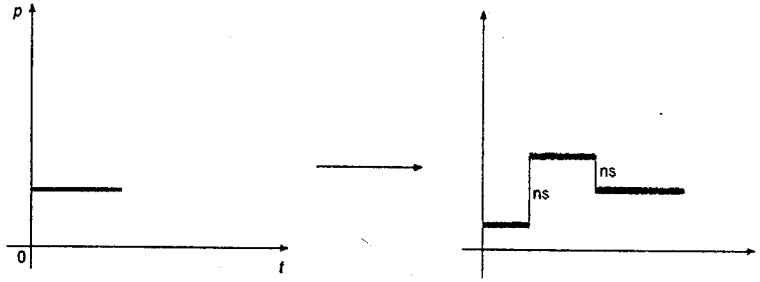


Fig. 2.20

The parts marked by *ns* of the curve, represent null sounds of zer duration.

A transformation of the type (Fig. 2.21)

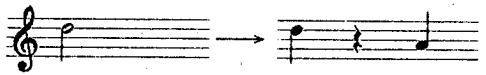


Fig. 2.21

is not topological when represented as (Fig. 2.22).

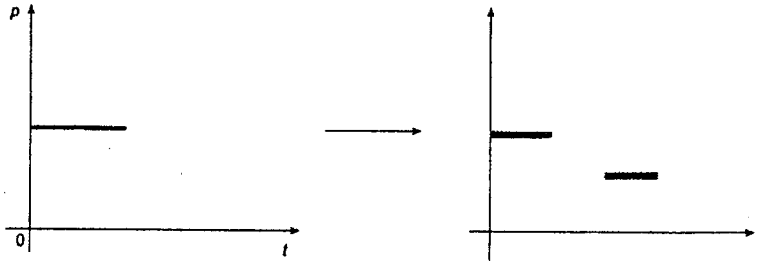


Fig. 2.22

But the same transformation is topological when represented as in Fig. 2.23

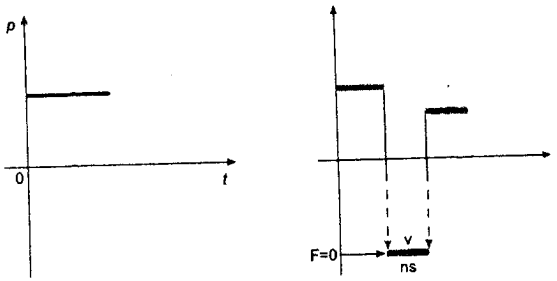


Fig. 2.23

where the segment marked with *ns* of the curve represents a null sound (the sound frequency is zero).

The set of forms the image of a sound may take in sound space is of power of continuum; hence, for the composer, this set is practically infinite.

But note: a transformation like that in Fig. 2.24a, b, c

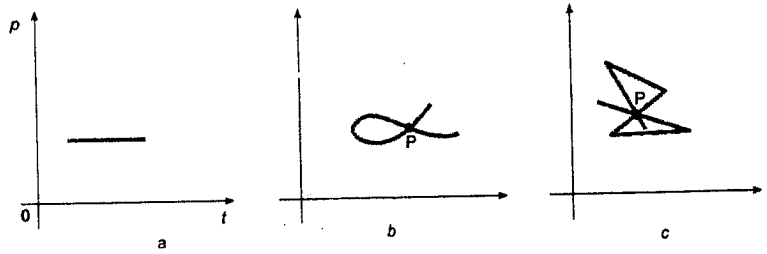


Fig. 2.24

is not topological since the one-to-one condition is not observed: different points of the initial form (Fig. 2.24a) correspond to the point *P* in the image obtained by transformation (Fig. 2.24b, c).

Also, it should be mentioned that if for the mathematician the image of the curve in Fig. 2.25 represents a sound from the *auditory* point of view, one *hears four sounds*, which, in musical notation, would appear as in Fig. 2.26.

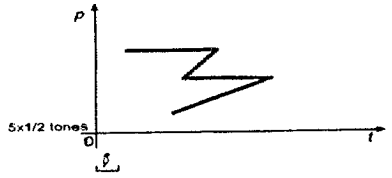


Fig. 2.25

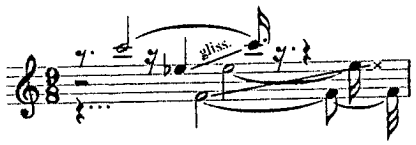



Fig. 2.26

This is due to the fact that one perceives simultaneously sounds c pitches of different values corresponding to a certain interval defined on th time axis.

Thus, in order to transform auditorily a sound represented by a curv C_1 into another and *only one other sound*, all the points of the curve C (the image through some homeomorphism of C_1) should have differer abscissae.

2.3.2. Transformation of a sound in musical notatio and the corresponding operation in sound spac R^2. Definition vector. Transformation vector

Let us assume that a composer desiring to begin a work would sta with Fig. 2.27a.

After a certain deliberation, he would change the musical text int Fig. 2.27b, and then into Fig. 2.27c.

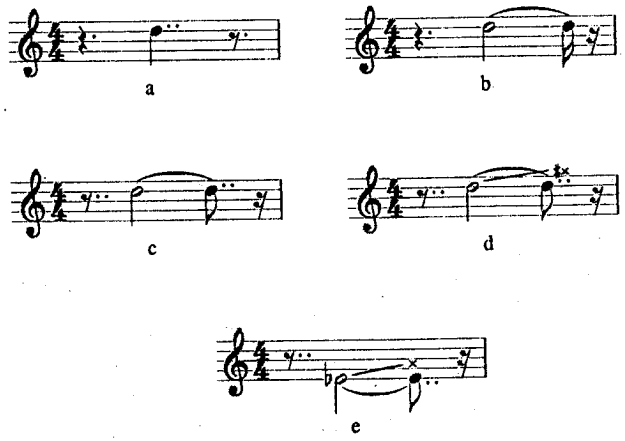



Fig. 2.27a,b,c,d,e

Considering the new image, he then decides that the sound should n have a constant pitch all along its duration and that he would prefer to ha a sound starting from the same pitch, but as ascending glissando of a augmented fourth (Fig. 2.27d).

Finally, the composer thinks it better that the work should start in anoth register and he transposes the sound a major seventh lower (Fig. 2.27e).

And so, by several changes made on the initial image, the followir transformation was obtained (Fig. 2.28).

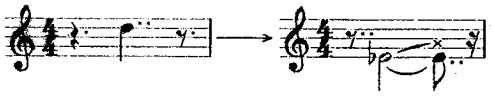


Fig. 2.28

This homeomorphism, represented in sound space $\mathbf{R}^2$, has the form (Fig. 2.29)

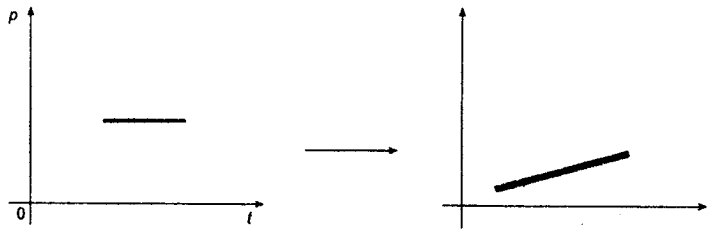


Fig. 2.29

Actually the composer has transformed the initial sound (S_1) by modifying the coordinates of the points A and D (2.1.1), more exactly by transforming:

(1) the values of the appearence and disappearance data (tA, tD),
(2) the values of the sound pitch at these moments (pA, pD).

This operation may be written down as:

$S(tA, pA, tD, pD) \rightarrow S'(tA + m,\ pA + n,\ tD + q,\ pD + r)$ where:

m, n, q and r are parameters that may take positive, negative or zero values. $S(tA, pA, tD, pD)$ is the *definition vector* (Dv). The m, n, q, r parameters also represent a vector which we will refer to as *transformation vector* (Tv).

The $S \rightarrow S'$ transformation is therefore obtained by adding the vectors Dv and Tv.

The definition vector of the sound S_1 (Fig. 2.27a) has the form $Dv(S_1) =$ $= 12\ \ 14\ \ 26\ \ 14$.

In order to obtain the transformation in Fig. 2.29, the variables m, n, q, r components of vector Tv will have the values -5, -11, 4, -5, respectively. The definition vector of the sound S_5, (Fig. 2.27e) $Dv(S_5)$ is represented by the sum $Dv(S_1) + Tv$

$Dv(S_1)$	12	14	26	14	+
Tv	–5	–11	4	–5	
$Dv(S5)$	7	3	30	9	

We leave it to the reader to determine, if he/she desires, the transformation vector leading the sound S_1 to the sounds S_2, S_3, S_4.

2.3.3. Augmentation and diminution of a musical pitch interval represent topological transformations in sound space $\mathbf{R}^2$

From a topological point of view, all pitch intervals, each being defined in the sound space $\mathbf{R}^2$ by the ordinates of a couple of points (2.1.4), represent a class of equivalence, since each couple can be viewed as being the transformed image of another couple considered the "initial form" (A.1.1). The set of elements of a class of equivalence being of the power of continuum, the set of musical intervals that may be imagined by the composer in the continuous space $\mathbf{R}^2$ is as a matter of fact, infinite. From a topological point of view, the intervals (P_1, P_2) in Fig. 2.30 are equivalent.

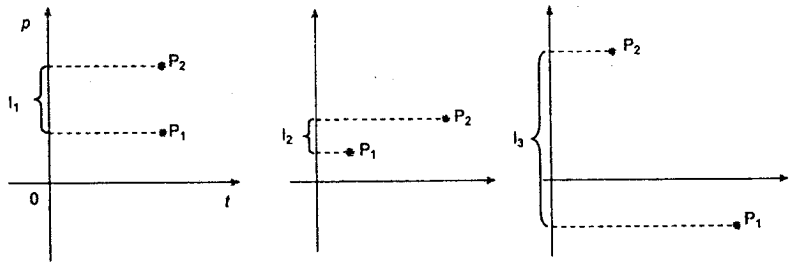


Fig. 2.30

We recall that the laws of traditional harmony use the method of "augmenting" and "diminishing" a pitch interval by "altering" one or both sounds forming the respective interval (1.5.6).

A pitch interval I_1 being given, represented by the points P_1 and P_2 of ordinates m and n respectively, the "augmentation" or "diminution" of I_1amounts to its transformation into an I_2 interval, the transformation being expressed by the relationships:

I_1: $y(P_1) = m$ $\qquad$ I_2: $y(P'_1) = m \pm 1$
$\quad y(P_2) = n$ $\qquad\qquad\quad$ $y(P'_2) = n \pm 1$

Consequently we may state: *In sound space $\mathbf{R}^2$, the augmentation or diminution of a pitch interval amounts to a topological transformation expressed by the transformation of the ordinate values of the sound points defining the interval.*

2.3.4. Polyphony considered as families of topological transformation of a curve representing a "cantus firmus" (1.5.5)

Definitions

Monody is, in the most general sense, a musical fragment performed by a single "voice".

Polyphony is a superposition of distinct monodies simultaneously evolving in time, but with a certain interdependency.

In the following, we may state a formalisation of the previous definitions.

1. We call *monody* a sound form containing a linear sound event V or a sequence of such elements that follow one another in time, each one being represented by a curve C of equation $y = f(x)$, where $1x=t$ and f is a continous function (Fig. 2.31).

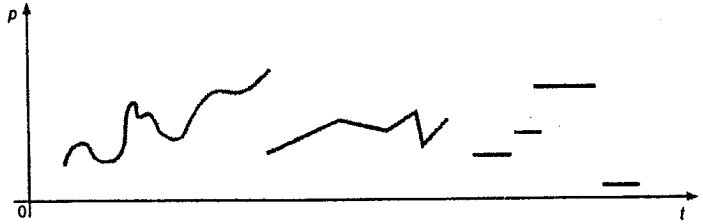


Fig. 2.31

2. We call *polyphony* a sound form consisting of two or more linear sound events, say $V1, V2, ..., Vn$, which with few exceptions satisfy

$$y(Vi) \neq y(Vj) \qquad i, j = 1, ..., n \qquad i \neq j$$

and in order to define the concept of "simultaneity", each point ai, bi, ci, $i = 1, 2, ...n$

$$a1, b1, c1 \in V1$$
$$a2, b2, c2 \in V2$$
$$an, bn, cn \in Vn$$

must satisfy the following conditions:

$x(a1) = x(a2), ..., xan)$; $x(b1) = x(b2), ..., x(bn)$, etc. (Fig. 2.32).

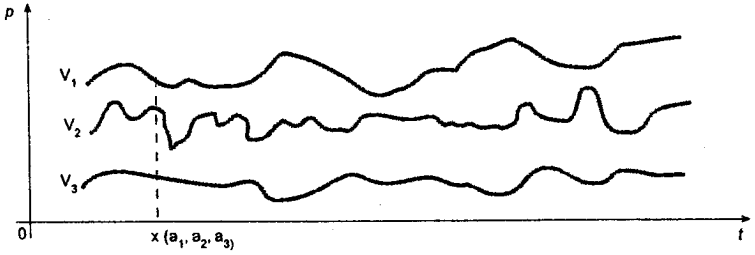


Fig. 2.32

The curves representing musical sounds are as a rule of the form $y = p$ (pitch), $a \leq t \leq d$ (rectilinear sounds of constant pitch). The polyphony ("note-to-note" (1.5.5)) appears as a set of homeomorphic images obtained by transforming an initial "cantus firmus" sound form (1.5.5) (Fig. 2.33a,b).

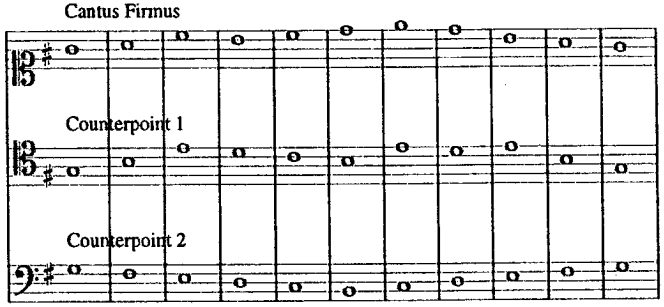


Fig. 2.33a

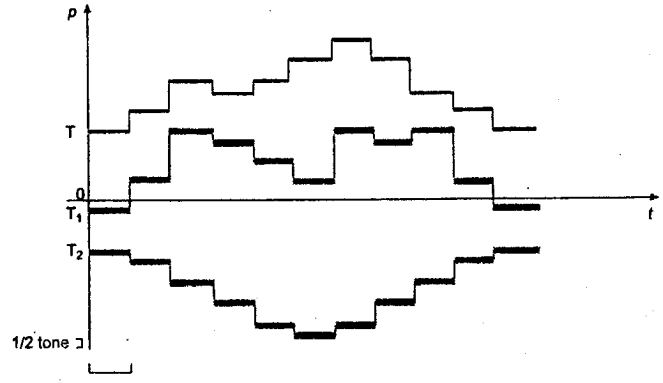


Fig. 2.33b

However, polyphony is not limited only to the form "note-to-note". As is well known, there are polyphonies where, to a sound of a certain duration and

pitch belonging to "cantus firmus", several sounds of different durations and pitches (Fig. 2.34a) sometimes interpolated with "null sounds" ("rests") (Fig. 2.34b) are associated. In the end, to a certain number of sounds of different pitches (in traditional music these sounds are usually characteristic for a certain key "function" (4.2.12)), another number of sounds of different pitches having the same "function" are associated (Fig. 2.34c).

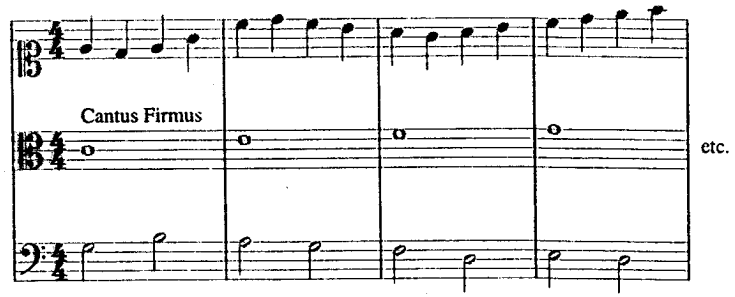



Fig. 2.34a

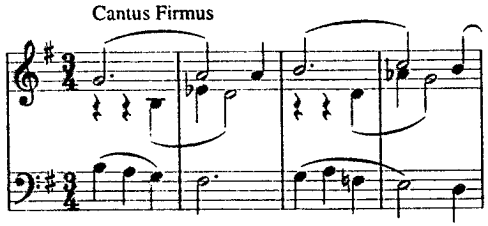


Fig. 2.34b

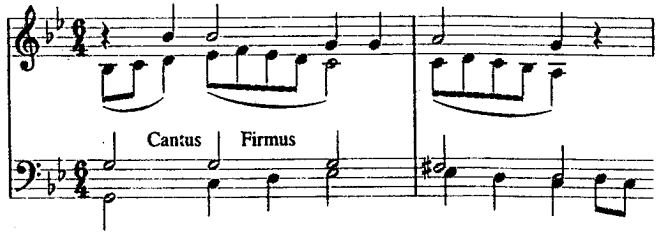


Fig. 2.34c

Resorting to the concepts of null sound events (1.1.3) (musical sounds, in our case) which at certain moments *tx* have zero duration or, during certain time intervals, the number of oscillations of the sound source (frequency) (2.3.1) is zero, we may define polyphony by a union of "stepped curves". In this sense, the polyphony in Fig. 2.34a may be interpreted as a family of homeomorphic curves representing sounds

having at certain moments, zero duration (denoted by *ns* – null sounds) (Fig. 2.35a), and the polyphony in Fig. 2.34b may be interpreted as a family of homeomorphic stepped curves representing sounds which during certain time intervals have zero frequency (Fig. 2.35b).

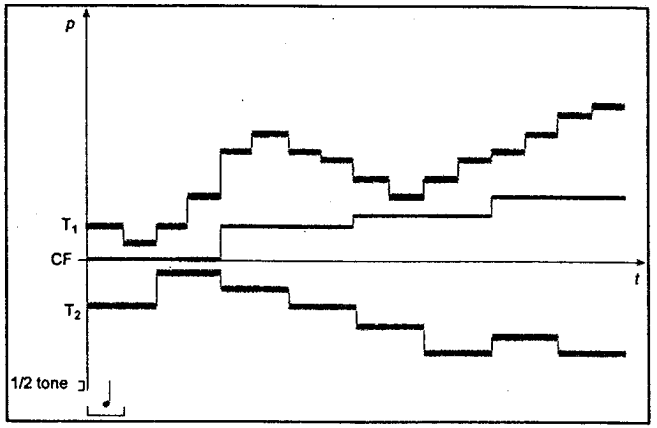


Fig. 2.35a

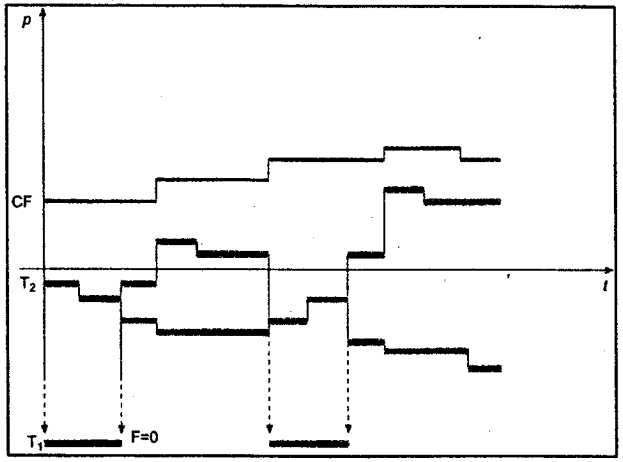


Fig. 2.35b

Figs. 2.33a,b; 2,34a,b,c and 2.35a,b prove visually the definition introduced under 2.3.4 i.e., that *a polyphony can be interpreted as a set of topological transformations of a given curve V, so that a given point in V corresponds to another point in V', if, and only if, they have the same abscissa.* In this interpretation, the transformational idea included in the notion of "Counterpoint", deriving from the Latin "*punctus contra punctum*", may be

explained, as mentioned under (1.5.5; 2.3.4). "Cantus firmus" is the original monody, and the "counterpoints" (1,2,...n) are the transformed forms of the "cantus firmus" under the above mentioned conditions.

2.3.5. The mathematician-composer of the 11th century

It is fascinating to remark the composer's mathematical intuition around the year 1100. Almost five centuries before the idea of a two-dimensional space made its appearance in mathematics and many centuries before the concept of function, a spatial representation and a topological transformation were enunciated, composers of those times wrote a polyphony as shown in Fig. 2.36:

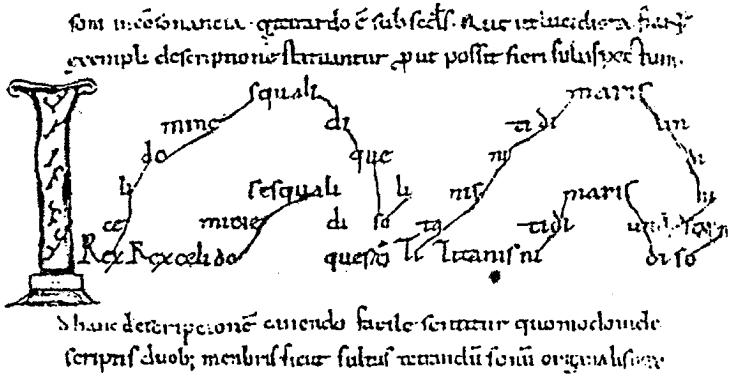


Fig. 2.36

The similitude between the representation in Fig. 2.36 and the graphical representation of a function and its topological transformation is striking.

2.3.6. Heterophony considered as families of topological transformations of a curve. Transformations bound to observe certain specific restrictions

Heterophony is a superposition of similar monodies with a simultaneous evolution which keeps them dependent on each other, yet without being identical.

In order to describe the heterophony phenomenon of two sounds, we have first to recall the concept of distance between two curves which are obtained from one another by topological transformation. Let C_1 and C_2 be two such curves and let Γ_1 and Γ_2 be their images. Let h be a homeomorphism of the specified type, between Γ_1 and Γ_2. Let P_1 be a point in Γ_1 and $P_2 = h(P_1)$ the associated point through h in Γ_2. We will consider the number

$$d(C_1, C_2) = \inf(\sup)\ \mathrm{dist}(P_1, h(P_1))$$
$$h \in H$$
$$(P_1, h(P_1))$$

where H is the set of all homeomorphisms of the specified type, dist(P_1, $h(P_1)$) being the Euclidean distance from point P_1 to point $h(P_1)$; *sup* represents the upper boundary and *inf* the lower boundary.

The number $d(C_1, C_2)$ is referred to as the distance between C_1 and C_2. It is known that $d(C_1, C_2) > 0$; if $d(C_1, C_2) = 0 \Leftrightarrow C_1 = C_2$; $d(C_1, C_2) = d(C_2, C_1)$; $d(C_1, C_2) = d(C_1, C_3) + d(C_3, C_2)$. Regarding the two curves C_1 and C_2, of a lower distance from the positive number ε, we may adopt the following image: the graph of C_2 does not leave a casing 2ε wide enclosing the C_1 graph and is equally distributed on either side of this graph. Apart from this restriction, the C_2 is completely free to turn around within this casing. When C_1 and C_2 are sounds in the sound space, the phenomenon described above is the heterophony phenomenon in musical theory, but here we will refer to it as ε-heterophony in order to emphasize the ε parameter which is the heterophony "amplitude". The nearer to zero the positive number ε, the smaller the heterophony amplitude, hence the heterophony becomes less perceptible. When $\varepsilon \to 0$, the heterophony tends to monody.

As in the polyphony case, the heterophony may be plotted by several curves in the $\mathbf{R}^2$ sound space (Fig. 2.37):

Fig. 2.37

Just the same as in the polyphony case, heterophony may be made up by several stepped curves.

It is easily noticed that heterophony may be interpreted as a set of topological mappings of a curve in $\mathbf{R}^2$, transformations that observe the restrictions above.

2.3.7. Transformation of musical sound forms within the composition method. So-called "variations on a certain theme", interpreted as topological transformations

Recalling 1.1.4, 5 and 1.2.1, 1.2.2, musical motifs or themes and, in fact, musical fragments of any length, may be represented in sound space as "sound forms".

We can easily notice that all sound forms obtained by topological transformations from an initial sound form considered "motif" or "theme" constitute a class of equivalence.

Topologically, the sound forms in Fig. 2.38 and their image in musical notation, Fig. 2.39, are equivalent.

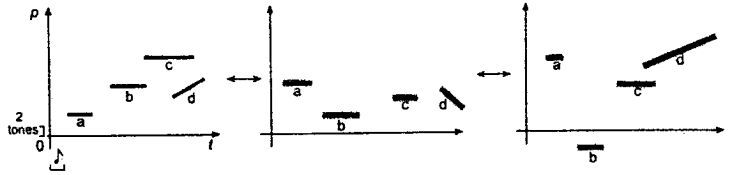


Fig. 2.38

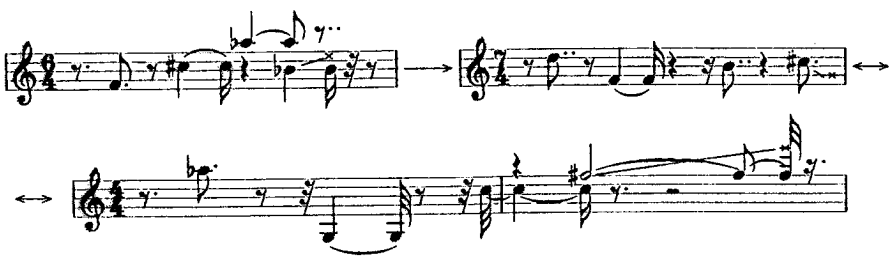


Fig. 2.39

Consequently, the "variations" of some motifs or themes from known musical literature, represented in the $\mathbf{R}^2$ sound space, may be interpreted as particular cases of topological transformations. The specific aspect of these homeomorphisms consists in the fact that the transformation should be made so that the obtained images keep, although sometimes quite vaguely, the musical sense of the initial form.

An example would be the transformation of a theme, initially conceived in a "major" key, into a homonymous "minor" key. Such a method, used by Gustav Malher in his Ist Symphony for the theme of the famous "Frère Jacques" canon, is illustrated in Fig. 2.40a (in the first two bars, Mahler also transforms the sound A2 into D3).

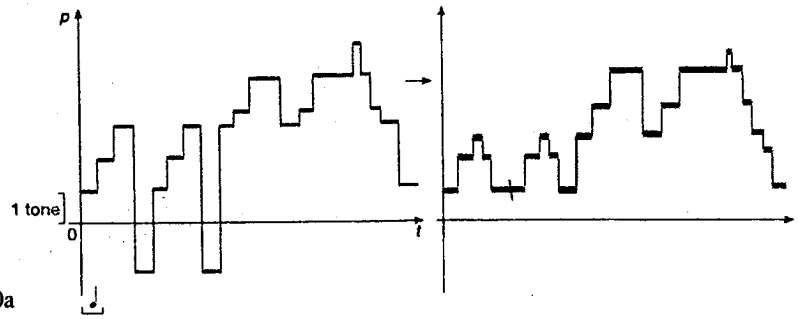


Fig. 2.40a

Represented in the $\mathbf{R}^2$ sound space, this transformation reveals itself as in Fig. 2.40b.

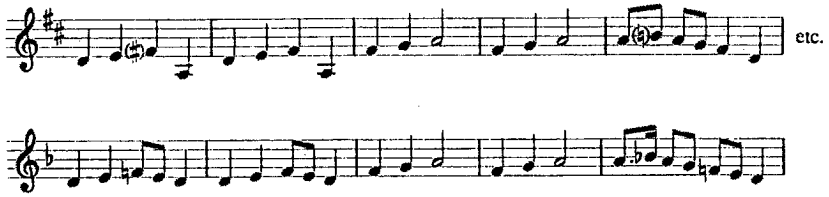



Fig. 2.40b

Another example: the well known theme of Mozart's piano Sonata in A major (Fig. 2.41)

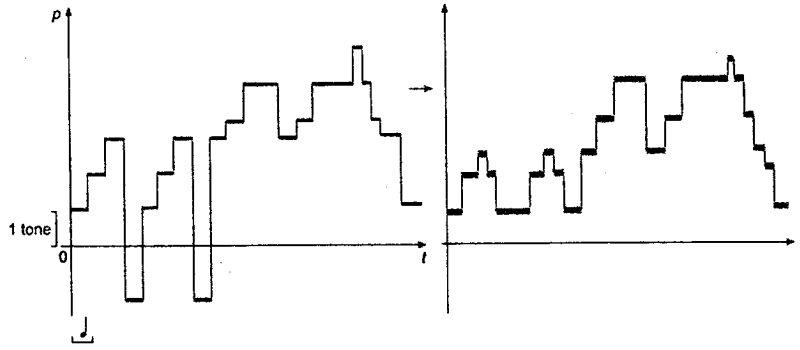


Fig. 2.41

Fig. 2.41a

Making use of the concept of non-sound (frequency zero (2.3.1)), representing "rests", Fig. 2.41a shows the beginning of the original theme and its three "variations" viewed as topological transformations of a curve in $\mathbf{R}^2$.

2.3.8. The "modulation", a particular case of topological transformation used in music

Within the tonal traditional music composition process, we refer to as "*modulation*" the transition from a given key to another one.

As we know, keys are modes of 7 sounds. According to (2.2.2) the set of modes of n sounds is isomorphous with a class of equivalence of topological spaces with a modal structure of n non-null sound zones. It is easy to notice that *modulation is a topological transformation of a space with a certain modal structure into another space of topologically equivalent modal structure.*

In other words, a modulation appears as a homeomorphism transforming the topology of a certain space into another equivalent topology (Fig. 2.42).

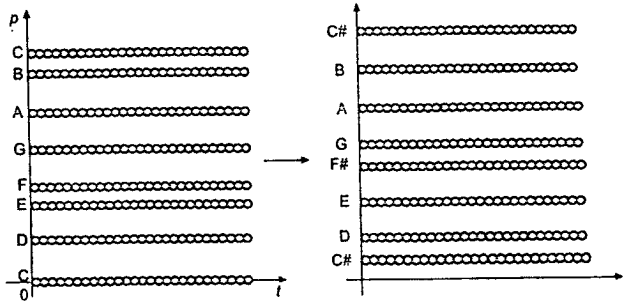


Fig. 2.42

2.3.9. Modal adjustement (*MA*): transformation of a sound form defined in R^2 continuous sound space, into a sound form defined in a space with a certain modal structure

Let F be a sound form defined in S_1 continuous sound space and let S_2 sound space have the modal structure characteristic of the D major mode (Fig. 2.43).

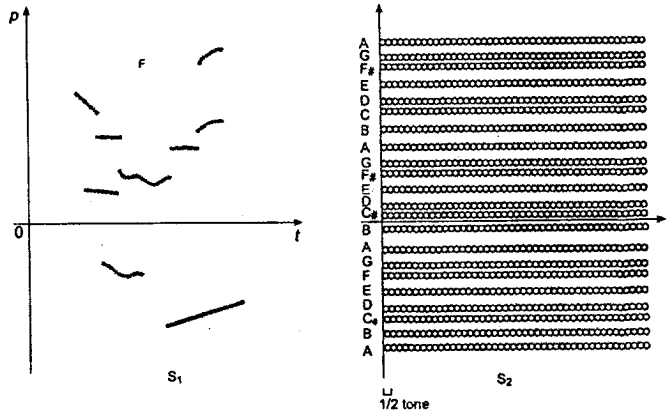


Fig. 2.43

Now, let h be a homeomorphism determining the correspondence of every sound event from S_1 to a sound in S_2, defined by the same abscissae, but of ordinate equal to the ordinate of the non-null sound zone having the nearest value to the ordinate of the moment of appearance of the considered event in S_1 (Fig. 2.44).

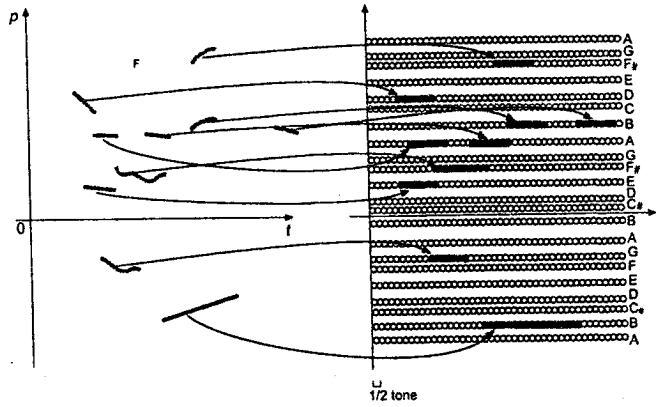


Fig. 2.44

We will refer to such a transformation as *Modal Adjustement (MA)*. Particularly, a musical sound form transformed by "alteration" of certain sounds, for instance the one represented in Fig. 2.45:

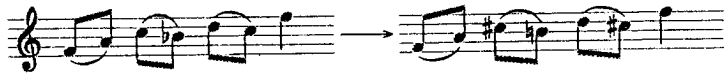


Fig. 2.45

may be interpreted as a "Modal Adjustement": a sound from a space with a certain modal structure is "adapted" to a space with a different modal structure (Fig. 2.46).

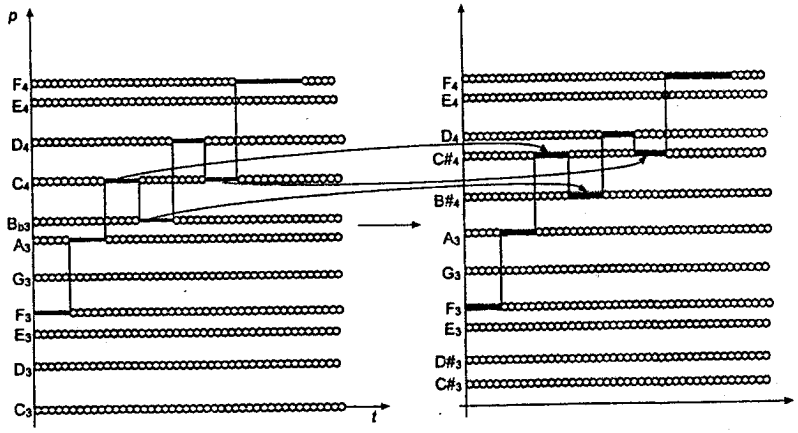


Fig. 2.46

As we will see, the methods of composition "imitation" and "reversion" of a theme frequently require Modal Adjustements.

2.4. TRANSFORMATIONS IN SOUND SPACE R^2 ENDOWED WITH AFFINE STRUCTURE

2.4.1. Transformation by translation

Transformation of a musical sound of a certain duration and pitch into a sound of the same duration and pitch, but of a different appearence moment, is a translation with respect to a vector parallel to the time axis (Fig. 2.47a, b).

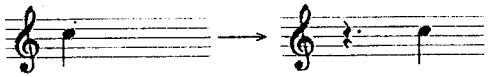


Fig. 2.47a

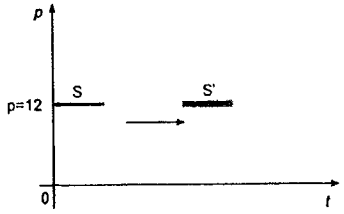


Fig. 2.47b

Transformation of a musical sound of a certain duration and pitch into a sound of the same duration and of the same appearence moment, but of different pitch, is a translation with respect to a vector parallel to the pitch axis (Fig. 2.48a, b).

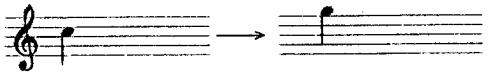


Fig. 2.48a

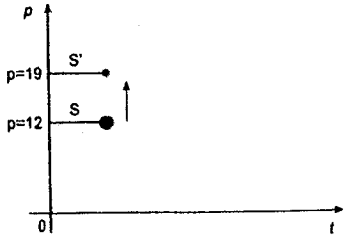


Fig. 2.48b

Transformation of a musical sound of a certain duration and pitch into a sound of the same duration, but having another appearance moment and another pitch, is a translation with respect to a vector defined by two points marking the appearance moment and pitch of the initial sound and the appearance moment and pitch of the transformed sound (Fig. 2.49a, b).

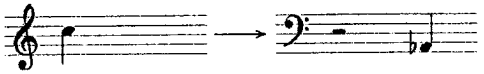


Fig. 2.49a

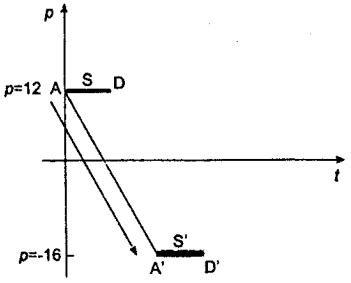
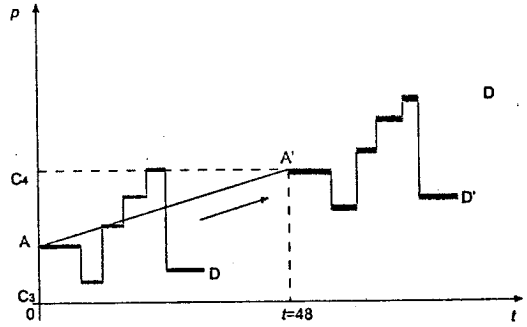


Fig. 2.49b

"Imitations" of a musical motif on various steps of a mode is a translation related to a vector defined by the coordonates of two points A and A' representing, respectively, the appearance moment and pitch of the first sound of the considered motif and the appearence moment and pitch of the first sound of the transformed motif.

Fig. 2.50a

Fig. 2.50b



Figs. 2.50a and b illustrate a sound form and its "imitation" a fifth higher in musical notation and interpreted as topological transformation through translation.

"Imitations" with Modal Adjustment (MA) (2.3.9) *interpreted as a composition of translations*

In the previous example, the "imitation" allows an identical reproduction of the initial form. But often, when the composer wishes to

repeat a musical pattern on another step of the scale, the pattern cannot be identically reproduced since, generally, the intervals between the mode's various steps are different: a major second may correspond to a minor second, and reversely, a perfect fourth may correspond to an augmented fourth, and so on (Fig. 2.51).

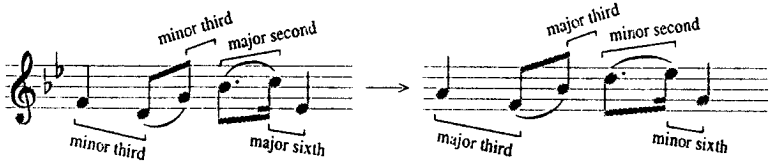


Fig. 2.51

This "imitation" a major third higher brought about in the $\mathbf{R}^2$ sound space, is the result of two transformations: *a translation composed with a Modal Adjustment.* The modal adjustment represents, in this case, the translation of certain sounds of the fragment (Fig. 2.49), so that they correspond to the "new" modal structure.

Figs. 2.52a and b illustrate the imitation as topological transformation obtained through a composition of translations.

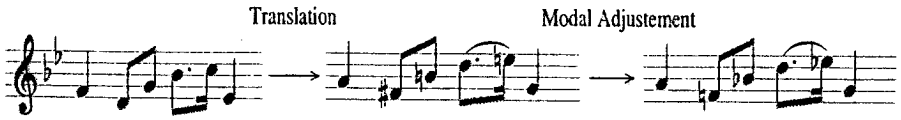



Fig. 2.52a

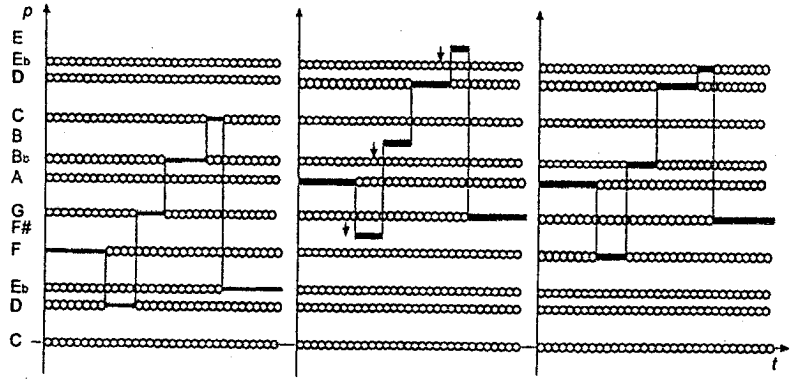
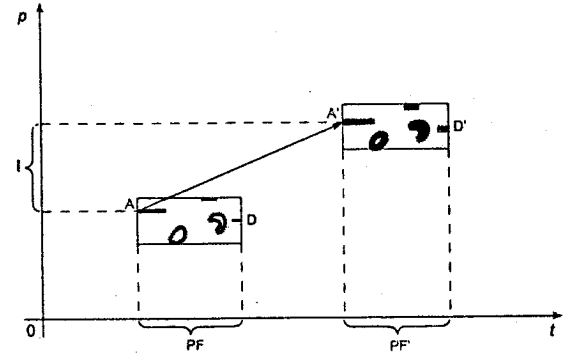


Fig. 2.52b

2.4.1a: *Various types of imitations, among which : repetition, "Basso Ostinato" canon and transposition, defined in terms of the interrelationships which can be established between the projections on the time axis of the initial forms and their transforms by translation*

Let A and D be the points representing the appearence and disappearence moments of the first and last sound, respectively, of a sound form SF, and let A' and D' be the corresponding points in the translated form SF'.[4] Let FP and FP' be the intervals defined by the projections of A and D and A' and D' points, respectively, on the Ox time axis and let I be the interval defined by the projection of the points A and A' on the *Oy pitch* axis (Fig. 2.53).

Fig. 2.53



Depending on the relationships that may be established between the intervals PF, PF' and I, we notice four cases of translation having musical significance:

I) $PF \cap PF' = \varnothing \quad I \neq 0 \quad xD < xA'$

The intersection of the PF and PF' intervals is empty; the translation generates an imitation at a given time interval and on another step of the scale (the case in Fig. 2.53).

Below we show a famous example of the considered type of transformation:

[4] In order to facilitate the understanding of the figure, SF and SF' are represented by the spaces of the considered forms, i.e., by the rectangular convex space, as defined in (1.1.5).

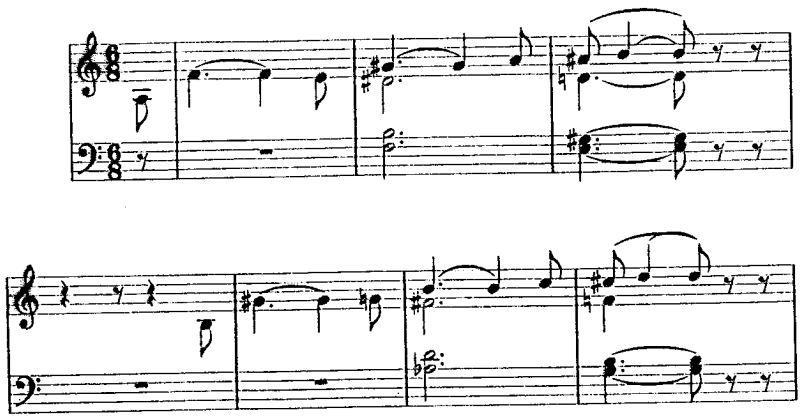



Fig. 2.54

R. Wagner, Prelude to *Tristan und Isolde*
(Imitation composed with Modal Adjustment)

II) $PF \cup PF'$ $I \neq 0$ $xD = xA'$

FP and *FP'* are united: translation generates concatenated imitation (Fig. 2.55a).

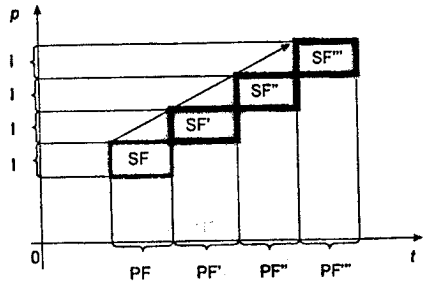


Fig. 2.55a

This operation can be repeated several times, as for instance in Figs. 2.55a,b.

Fig. 2.55b

Beethoven, IInd Symphony
(Imitation composed with Modal Adjustment)

If, in addition, $I = 0$, $(y(A) = y(A'))$, *repetition* of a fragment is obtained (Fig. 2.56).

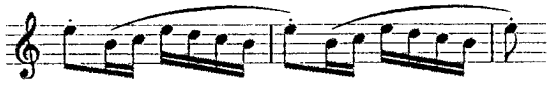
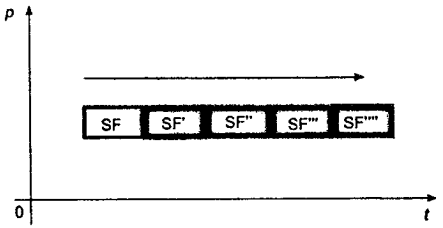



Fig. 2.56

Beethoven, VIIth Symphony

If, under the same conditions, this operation is carried out several times, the *"Ostinato"* musical pattern is obtained (Fig. 2.57a, b).

Fig. 2.57a



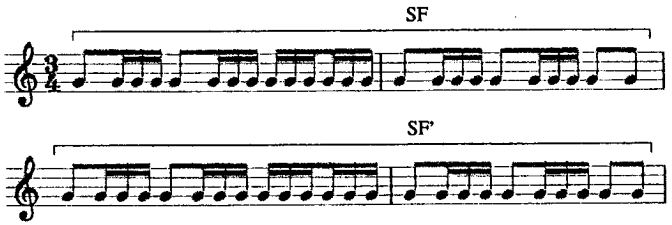


Fig. 2.57b

Ravel, Bolero

If the ostinato pattern is situated in the "bass" register, we are dealing with the "Basso Ostinato" pattern used to compose musical forms such as Passacaglia or Tarantella (Fig. 2.58).

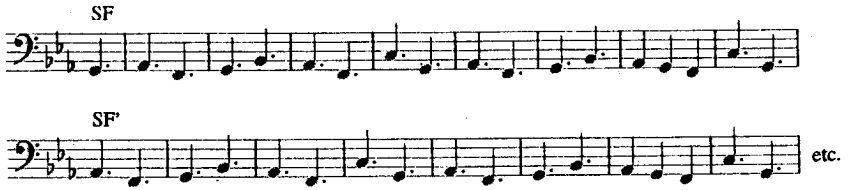



Fig. 2.58

Enescu, First Suite

III) $PF \cap PF' \neq \varnothing$, $I \neq 0$ $xD > xA'$

The intersection of the *FP* and *FP'* intervals is non-empty; the translation generates the musical form referred to as "Canon".[5] The starting moment of the imitating voice is defined by point A' abscissa; the musical interval when the imitation is carried out is defined as the interval *I* value.

The bringing about by translation of a canon at octave (Hindemith, "Mathis der Mahler") and of a double canon at higher fourth and lower fifth (Mozart, Requiem) is illustrated in Figs. 2.59a, b and 2.60a, b.

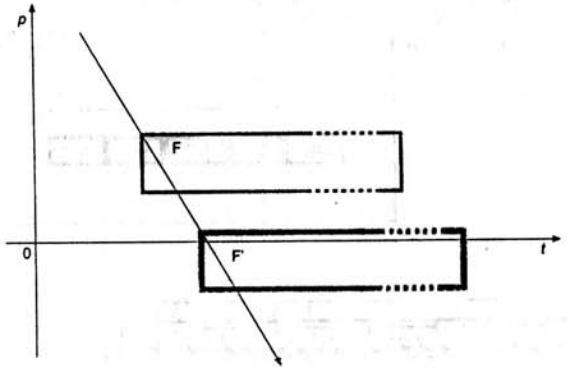


Fig. 2.59a

Fig. 2.59b

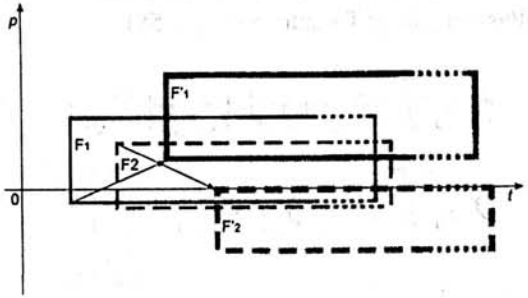


Fig. 2.60a

[5] The canon is a note-to-note imitation of a melodic line by one or several "voices" starting at different moments. The value of *I* defines the characteristic of the canon: canon at unison ($I = 0$), at third $I = 3$, octave or at any other interval ($I \neq 0$).

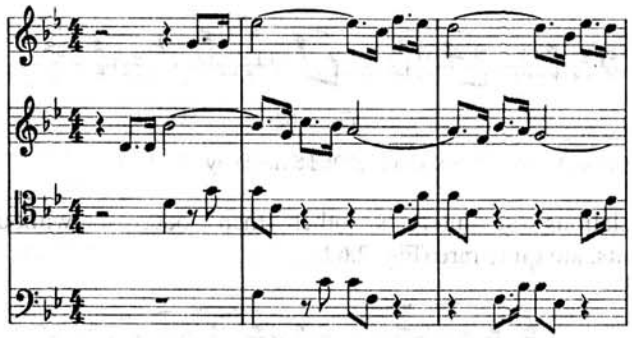



Fig. 2.60b

IV) $PF' \subset PF \quad I \neq 0 \quad xA = xA'$

The intervals *PF* and *PF'* are superimposed (Fig. 2.61a); translation generates:

a) *Transposition*[6]

The transposition of the fragment *SF* a minor second lower and an augmented fourth higher is illustrated in Fig. 2.61b.

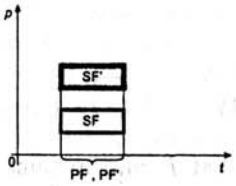


Fig. 2.61a

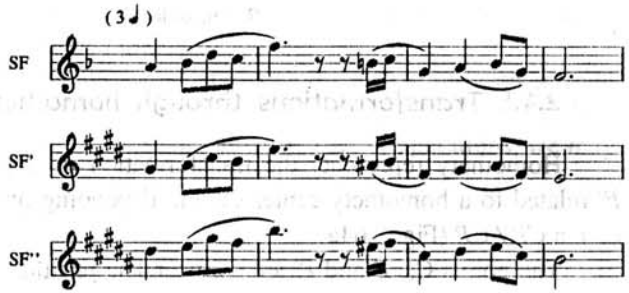



Fig. 2.61b

b) *Doubling of a melodic line at different intervals*[7]

When doubling occurs within a certain key (except doublings to octave), as a rule, the translation is composed with a modal adjustement (*MA*) (2.3.9) (Fig. 2.62).

[6] As known, by transposition is understood the transfer of a musical fragment or of a musical piece from a key to another one.This method is used in composition (a "theme" is written in various keys), as well as in performance (a song written for tenor is transposed for baritone).

[7] In the composition process, especially the orchestration one, the simultaneous reproduction of a melodic line at another pitch is usual.

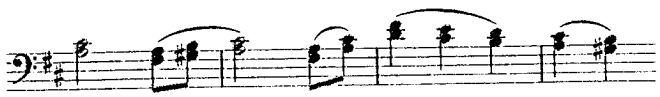



Fig. 2.62

Brahms, 2nd Symphony

Translations at intervals other than octaves, without modal adjustements, are quite rare (Fig. 2.63).

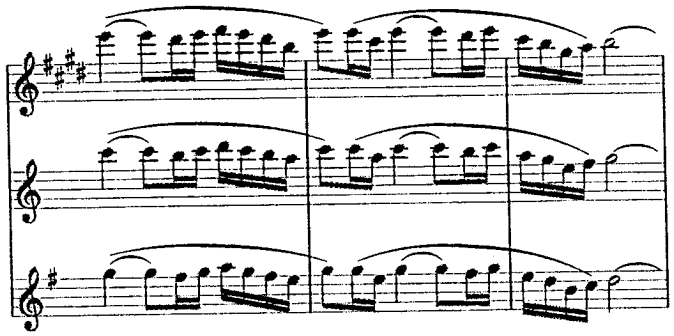


Fig. 2.63

Ravel, Bolero

2.4.2. Transformations through homothety

Homothety represents the transformation of a point P into a point P' related to a homothety center C_o and depending on the homothety ratio q, $q = CP'/CP$ (Fig. 2.64a).

The points C_o, P and P' are colinear.The position of P' with respect to the center C_o and the point P depends on the value of q (Fig. 2.64b).

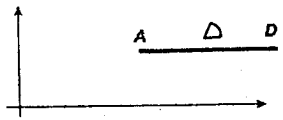


Fig. 2.64a

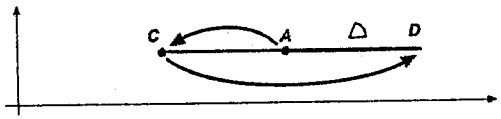


Fig. 2.64b

For $q = 0$, P' is located in the homothety center C_o. For $0 < q < 1$, P' finds itself between C_o and P. For $q = 1$, P' overlaps P (identical transformation, P is transformed into itself). For $- < q < 0$. P' is located between the homothety center and –. For $1 < q < +$ P' finds itself between P and +.

Remark

For $q = -1$ P' is symmetric to P with respect to the homothety center C_o.

2.4.2a: *The duration of a musical sound may be expressed in terms of a homothety center C_o and a homothety ratio q*

Let P and P' represent, respectively, the moment of appearence (A) and the moment of desappearence (D) of a musical sound.

The duration Δ of the considered sound may be expressed by the value of a homothety ratio q and a homothety center C_o located on the straight line defined by the points *A and D* (Fig. 2.64b).

$$(1)\ q = \frac{\overline{CD}}{\overline{CA}}$$

$$(2)\ \Delta = \overline{AD}$$

$$(3)\ \Delta = \overline{CA}\,(k-1)^{8}$$

q can take values only within the $(1, +\infty)$ interval. Indeed, as mentioned before for $q = 1$ point D overlalps A. As a consequence $CD - CA = 0$ hence $\Delta = 0$; the sound's duration is zero.

If $q \longrightarrow \infty$, $D \longrightarrow \infty$: the considered sound tends to have an infinite length ($\Delta \longrightarrow \infty$).

Allowing for CA to be a constant value for a set of operations, we may write down

$$\Delta = f(q).$$

Numerical example

In order to represent in sound space R^2 the duration of a musical sound by a homothety ratio q, the abscissae of the homothety center C_o and of the appearance moment (A) have to be known and C_o must have the same ordinate as A.

Suppose we wish to express a sound time length Δ equivalent to a dotted-quarter-note (12 units of thirty-second-note). Assuming that C_o is situated "at the beginning" of the considered musical fragment and the CA interval is equivalent to the quarter-note rest (8 units of thirty-second-note),

[8] $CD = k * CA$; $AD = CD - CA$; $AD = (k * CA) - CA$; $AD = CA(k-1)$
$AD = \Delta$; $\Delta = CA(k-1)$

from (3) $q = \frac{\Delta}{CA} + 1$ $\qquad q = \frac{12}{8} + 1 = 2.5$

In a similar way a duration Δ equivalent to a thirty-second-note added to a quarter-note (5 units) corresponds to a q value of $q = 5/8 + 1 = 1.625$.

The sound duration augmentation or diminution will correspond to a transformation through homothety of the D point (Figs. 2.65, 2.66).

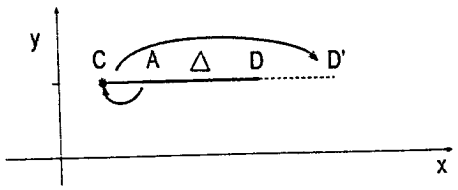


Fig. 2.65

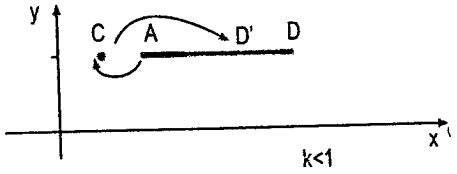


Fig. 2.66

Since q can take continuous values within the $(1, +\infty)$ interval, some of these values may not correspond to musical duration consisting of integer numbers of thirty-second units. If so, one should operate a round off as shown in 3.1.2.

Time retrograde mouvement

We recall that for q =1 the points A and D coincide; hence $\Delta = 0$.

For $q < 1$ the point D (representing the disappearence moment of a musical sound) inverses its sense with regard to the point A, evolving opposite to the time axis direction. Musical time begins to "grow" in the past. In a metaphoric acceptance, the duration Δ is "boring" in the zones of souvenirs.

According to (2.1.2), if represented by a vector, a sound of constant pitch will be represented in R^2 as in Fig. 2.67a.

←

A line segment AD repesenting a musical sound in retrograde movement will be notated as AD.

Figure 2.67b shows the image of a sound in retrograde time movement:

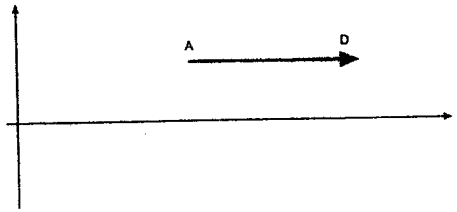


Fig. 2.67a

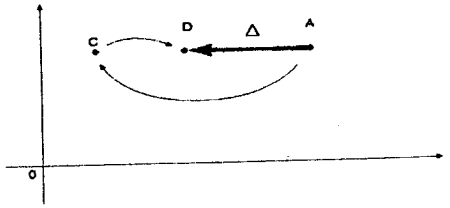


Fig. 2.67b

Remark

If the initial sound is a "crescendo" or a "diminuendo" one, the inversion of its direction induces the inversion of the dynamic characteristic: a crescendo sound becomes a diminuendo one and vice versa (2.3.3) (Fig. 2.68).

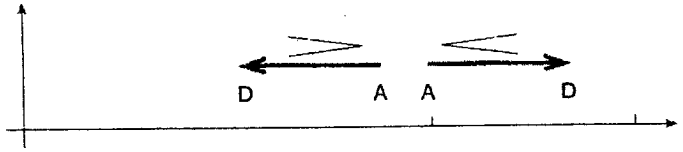


Fig. 2.68

As we will see in the next parts of this work, often transformations applied to sound forms induce change of direction of the respective sounds. An example is the transformation considered below.

2.4.3. A musical chord may be expressed in terms of a center and a family of homothety ratios

Since a musical interval may be expressed by a homothety ratio, and a chord is a family of intervals generated out of the same sound-origin, the chord may be interpreted as a family of transformations through *homothety* of a sound point, operated from the same center, but with different ratio values.

2.4.4. Reversion of chords expressed by homothety transformations

We here recall that reversion of a chord means its transformation through successive transpositions of the fundamental sound to octave (1.5.7).

Let us consider the points denoted by *Fs, T, F* and *S* (representing respectively the fundamental sound of a chord and its major third, perfect fifth and minor seventh) and the point C_0 of coordinates 0, –5, representing a homothety center. Since in semitones, the (*Fs, T*), (*Fs, F*) and (*Fs, S*) intervals have 4, 7 and 10 semi-tones, respectively, and, by using the collections of k ratios defined by means of the C_0Fs, C_0T, C_0F and C_0S distances, i.e.:

$$q = \{\frac{C_0F_s}{C_0F_s}, \frac{C_0T}{C_0F_s}, \frac{C_0F}{C_0F_s}, \frac{C_0S}{C_0F_s}\},$$

a dominant seventh chord and its three reversions may be generated out of the original sound F_s, by homothety of center C_0 and the above family of k homothethy ratios (Fig. 2.69).

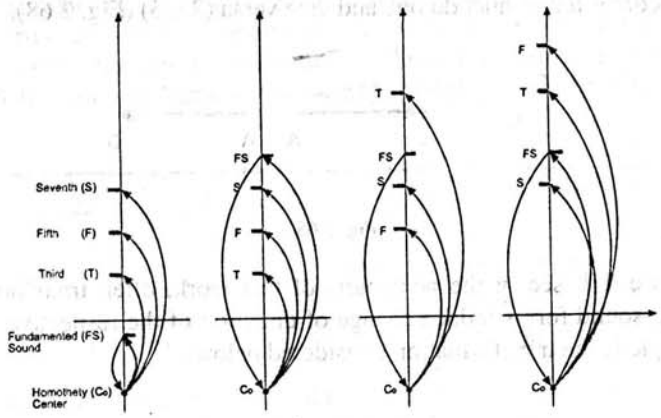


Fig. 2.69

If $FS = B2$, in musical notation, the obtained images will have the following form (Fig. 2.70).

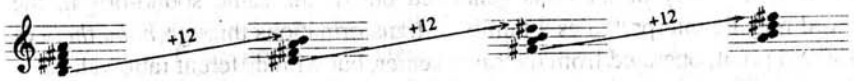


Fig. 2.70

2.4.5. Transformations through parallel homology

We invite the reader to examine A.3.6; A.4.2; particularly A.3.8 and A.3.13.

2.4.5a. *Augmentation and diminution of a sequence of sounds and rests: transformations through parallel homology*

Let us consider a sound form; its compression and expansion in the time axis direction may be obtained by parallel and orthogonal homology with respect to a perpendicular axis to the time axis. For $0 < k < 1$ the form is compressed; for $k > 1$ the form is expanded (Fig. 2.71). In the examples 2.71–74 the homology axis is supposed to be superimposed on the *Oy* pitch axis.

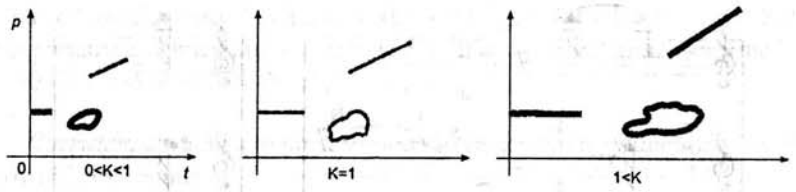


Fig. 2.71

The augmentation or diminution of a sequence of sound and rest durations by their doubling or bisecting or by the "added point" (1.5.8) amounts to the previous transformation where k has the 2, 0.5, and 1.5 value, respectively (Fig. 2.72).

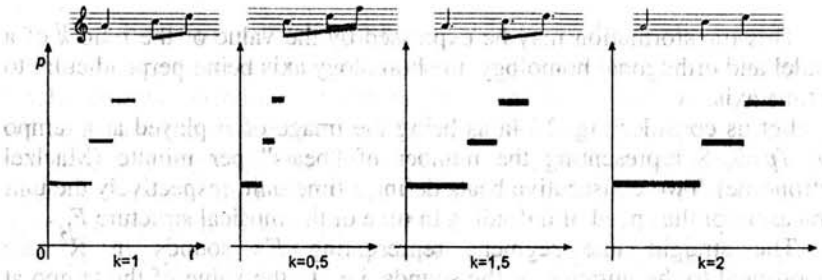


Fig. 2.72

2.4.5b. *Metronomic values* (1.5.9) *may be expressed by homology ratios*

Let F be a musical sound form (Fig. 2.73). Represented in sound space $\mathbf{R}^2$, the transformation of the "tempo" at which the sound form is performed implies the transformation of its image in $\mathbf{R}^2$.

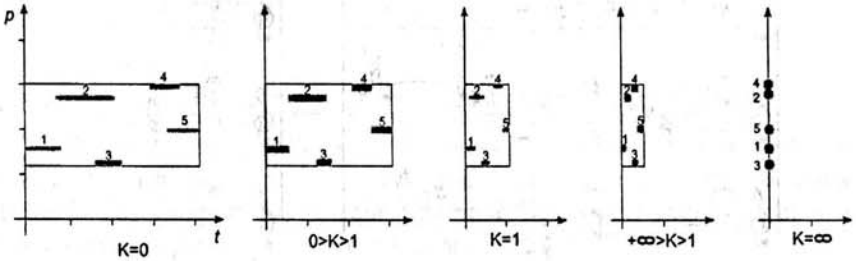


Fig. 2.73

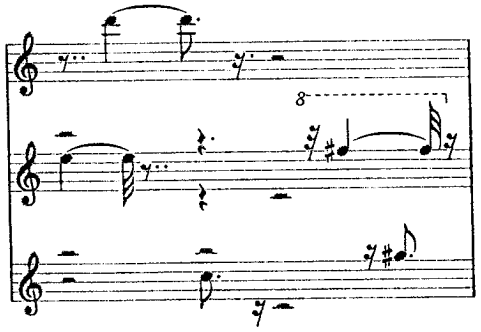



Fig. 2.74a

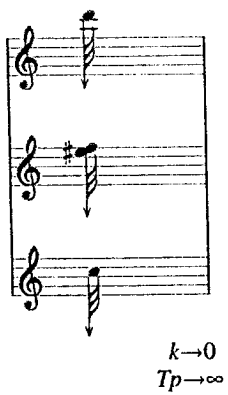



Fig. 2.74b

This transformation may be expressed by the value of the ratio k of a parallel and orthogonal homology, the homology axis being perpendicular to the time axis.

Let us consider Fig. 2.74a as being the image of F played at a tempo (Tp) Tp=\$, \$ representing the number of "beats" per minute (Maelzel Metronome). Two consecutive beats define a time unit, respectively the unit of measure of the speed of unfolding in time of the musical structure F.

The straight line segmens representing F's sounds in $\mathbf{R}^2$ are proportional to the duration of the sounds, i.e., to the value of the tempo at which F is performed.

It is easy to see that tempo values can be expressed in terms of k ratio values of parallel homology. The value of Tp is inversely proportional to the value of k: as the value of Tp increases, the value of k diminishes, and vice versa.

As far as the musical aspect of these transformations is concerned, $k = 1$ yields the initial image of F (Fig. 2.74a, Tp = \$). For \$ < Tp < ∞ the correspondent values for k homology ratio are: $1 > k > 0$. The sound form F will

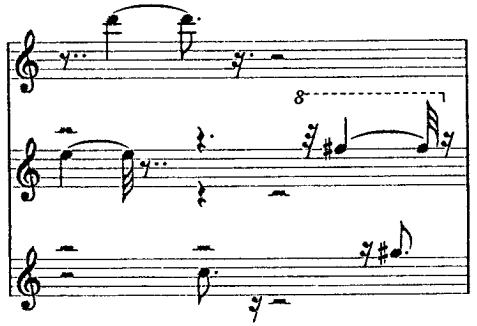



Fig. 2.74a

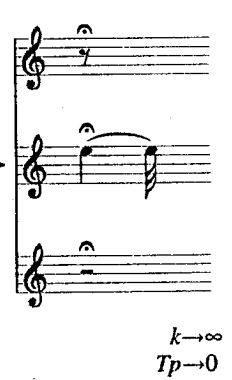


Fig. 2.74c

gradually "shrink in time" as Tp → ∞ (k →0) (Figs 2.74b). For 0 < Tp < \$ the correspondent k values are ∞ > k > 1. F will gradually "stretch in time" as Tp →0 (k → ∞) (Fig. 2.74c).

Remarks. Such a representation of metronomic values allows the following comments:

For k = ∞ (Tp=0), F will appear reduced to its first sound of infinite, endless duration.(Fig. 2.74c). For k=0, (Tp = ∞) all sounds of F will be lined up as "sound points" (1.1.3) on the homology axis. In musical terms, F becomes a "chord" of zero duration (Fig. 2.74b).

A Beethoven Symphony, or a Wagner opera played at Tp→ 0 or at Tp → ∞...

What holds for a small number of sounds, holds for the sounds of a symphony or of an opera. Beethoven's Ninth Symphony or Wagner's Nibelungenlied played at a tempo of value "neighbouring" zero (Tp → 0) will be reduced to a single sound of infinite time length. Played at a tempo of value "neighbouring" the infinite (Tp → ∞), the two masterpieces will appear as "chords" of infinitely small duration, containing all the sounds of the respective symphony or opera...

A person always in a hurry but still wanting to be able to tell that he/she has heard it, might prefer this mode of being a "connoisseur" in music...

2.4.6. "Ciphered bass" expressed by ratios of parallel and orthogonal homology

Let us consider two sounds S and $S1$ of different pitches and a homology axis @ parallel to the time axis; the pitch interval defined by the difference of the sound ordinates may be expressed in terms of a parallel and orthogonal homology ratio (Fig. 2.75).

$$k = \frac{MS_1}{MS}$$

This interpretation allows us to express the effectuation of a "ciphered bass" (1.5.5) by homology transformations. Let S be a sound of ordinate y belonging to a sound sequence constituting a "bass voice" segment and @ a homology axis parallel to the time axis of ordinate y_0 (Fig. 2.76).

Let *Cphi* (i=1, 2,..., n) be a collection of cyphers each cypher representing an interval within the musical scale: second, third, etc. (Fig. 2.76).

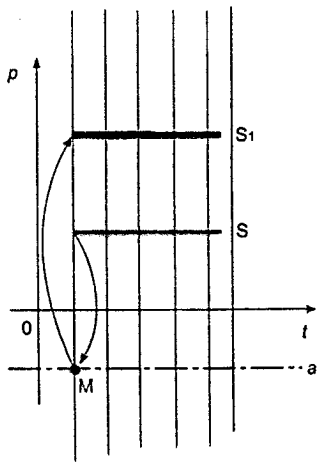


Fig. 2.75

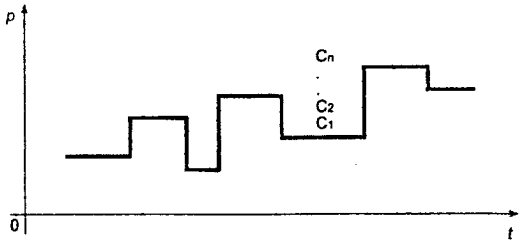


Fig. 2.76

By expressing the *Cphi* intervals in numbers of semi-tones we obtain the distances *SSi, Si* being the sound whose pitch is indicated by the corresponding cypher *Cphi.*

The @ and *S* ordinates being known, the sounds S_i ordinates (y'_i) are obtained from the equation defining the parallel homology

$y' = y_0 + k(y - y_0)$, $y = p$ (A.3.13) (Fig. 2.77).

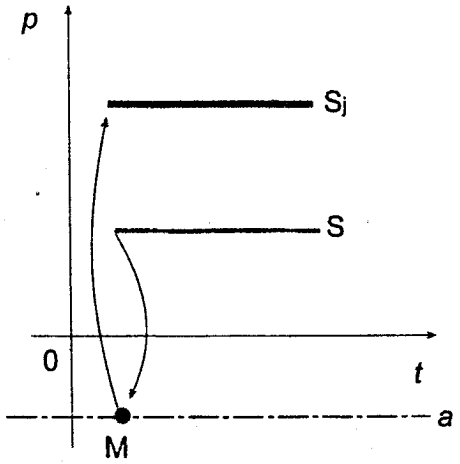


Fig. 2.77

Numerical example:

Let (Fig. 2.78a) be a ciphered bass and let (Fig. 2.78b) be its possible "bring about":

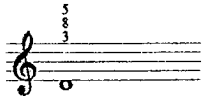


Fig. 2.78a

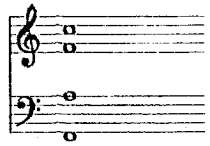



Fig. 2.78b

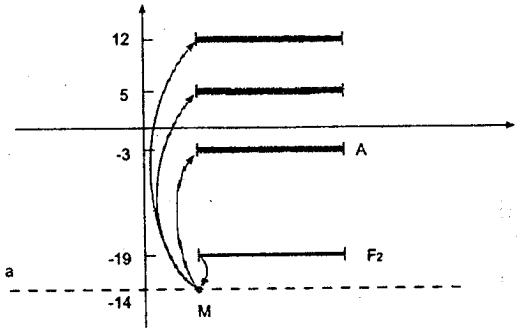
Let @ be a homology axis of equation $y = -24$ and let the sound F_2 of ordinate $y = 19$ be the considered "bass".

In order to express the pitch intervals represented by the ciphers 3, 8, 5 (third, octave, fifth) by homology ratios, let us consider C_1 = major third + octave = 4 + 12 = 16 semi-tones; C_2 = double octave = 24 semi-tones; C_3 = fifth + double octave = 7 + 24 = 31 semi-tones, values corresponding to the SS_1, SS_2, SS_3 distances, where S is the fundamental sound (F_2) and S_1, S_2 and S3 the sounds which define with respect to *S*, the musical intervals indicated by the ciphers 3, 8, 5 respectively. Through calculations, we find the

q ratios having the $\frac{21}{5}, \frac{29}{5}, \frac{36}{5}$ values, respectively, and the *yi'* values:

–3, 5 and 12 corresponding to the sounds A2, F3 and C4 (Fig. 2.79).

Fig. 2.79



2.4.7. Reversion, inversion, reversion of inversion and identical transformation of a sound form constitute a Klein commutative group structure (1.5.7). This group can be interpreted as an affine transformation subgroup in sound space R^2

We here recall the concept of commutative group structure.

An operation * induces in a non-empty set E a group structure, if there is a transformation of $E * E$ on E, i.e., for $a \in E$ we have $a * b \in E$

* is commutative $a * b = b * a$
* is associative $(a * b) * c = a * (b * c)$

there is a neutral element ε i.e., for $a \in E$, we have $\varepsilon * a = a * \varepsilon = a$

E is symmetrizable: for $a \in E$ there is $a^{-1} \in E$, so that $a * a^{-1} = a^{-1} * a = \varepsilon$.

A particular case of commutative group structure is the Klein group, defined by the following table:

*	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

The transformations through reversion (T_1), inversion (T_2), reversion of inversion (T_3), together (along) with the initial form (T_0) constitute a Klein commutative group also.[9] Indeed, when we compose two transformations, we obtain the following table, identical with the one above:[10]

0	T_0	T_1	T_2	T_3
T_0	T_0	T_1	T_2	T_3
T_1	T_1	T_0	T_3	T_2
T_2	T_2	T_3	T_0	T_1
T_3	T_3	T_2	T_1	T_0

This structure is easily seen in Fig. 2.80.

In the following we will refer to the group of these transformations as the group T.

The fact that the musical transformations: reversion, inversion and reversion of inversion, together with the identical transformation form a commutative group is, of course, interesting and proves once more the close connection between mathematical structures and the musical ones.

[9] See in this sense George van Hout also: Mathématique moderne, langage du futur. Ed. Nouwelaerts, Paris, 1970.

[10] It should be mentioned that these transformations form a group only when reversion and reversion of inversion are geometrical symmetries. This is obtained only in case the transformation is carried out in a modal space, whose steps are equidistant (for instance the dodecaphonic mode) or with respect to an axis dividing the space into two domains with symmetric structures (for example: the second step of major keys). In all the other cases, reversion is a geometrical symmetry composed with a "Modal Adjustment" (2.3.9). The problem is similar to that of "imitating a theme on the various steps of a scale".

*This **"a posteriori"** remark, however, does not enrich the inventory of composition methods. But, when we interpret the group T as being a subgroup of a larger group of transformations, there results as a corollary the fact that, by starting from the traditional principles of composition methods, many more creation tools belonging to the same family are now available to the composer. This is what we intend to do in the following.*

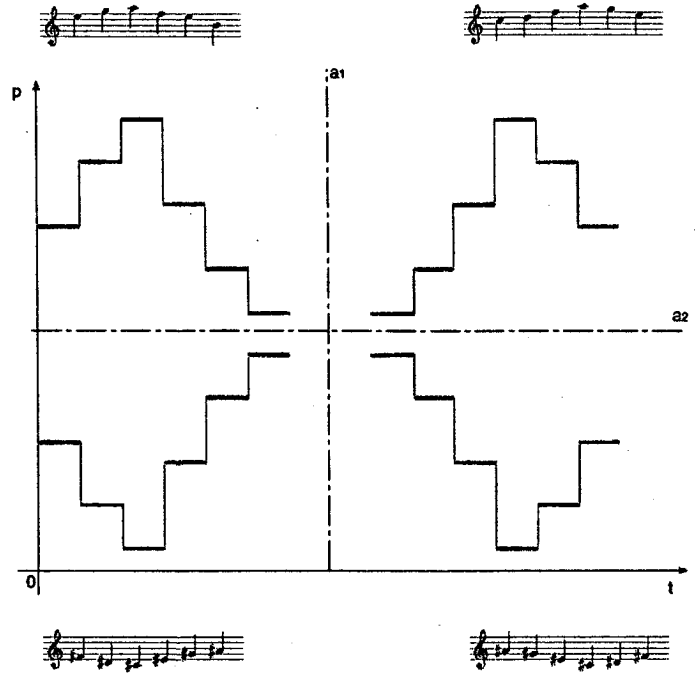


Fig. 2.80

2.4.8. Group *T* interpreted as a subgroup of the affine group of transformations. A musical theme reversion and inversion interpreted as transformations by parallel homology. Reversion of inversion interpreted as a transformation by homothety

It is shown in A.3.13 that the axial symmetry is a particular case of parallel and orthogonal homology. *Reversion*, being a symmetry with respect to a horizontal axis, can be interpreted as a transformation by homology if the homology axis is parallel to the time axis, the projections lines orthogonal and the homology ratio $k = -1$.

In the same sense, a theme *Inversion* is a transformation by parallel and orthogonal homology, on condition the homology axis is perpendicular on the time axis and $k = -1$.

Reversion of Inversion is a central symmetry (2.4.2), since the central symmetry is a homothety with $q = -1$ ratio (A.3.15).

Finally, parallel and orthogonal homology with k = 1 gives an identical image.

Since reversion, inversion, reversion of the inversion and identical transformation (group T) may be interpreted as particular cases of homologies and homotheties, and since these are transformations in affine space, it follows that the group T may be interpreted as a subgroup of affine transformation. Fig. 2.81 synthetically shows the group T as a subgroup of affine transformations.

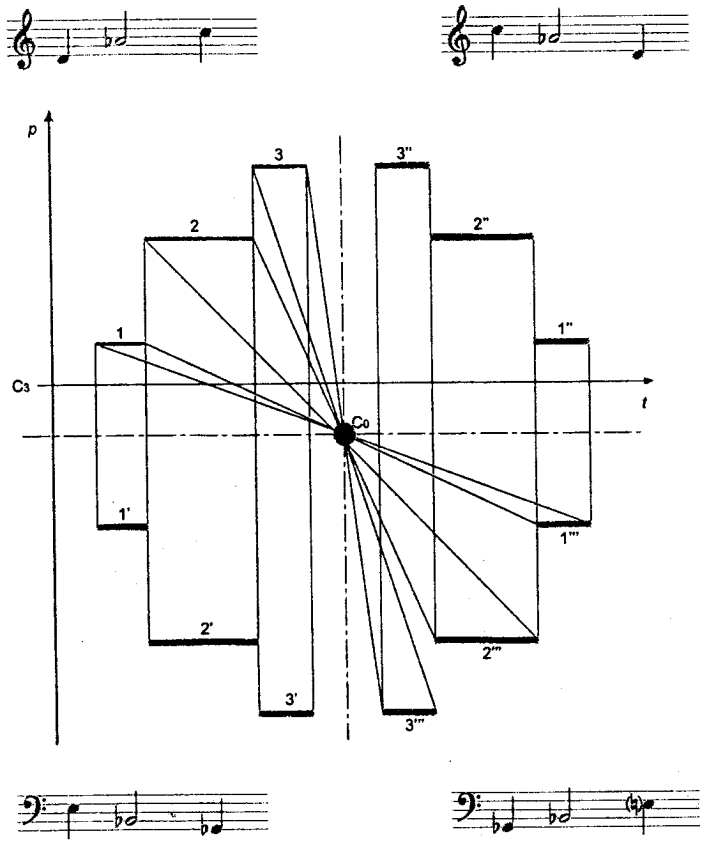


Fig. 2.81

2.5. TRANSFORMATIONS IN SOUND SPACE R^2 ENDOWED WITH EUCLIDEAN STRUCTURE

2.5.1. Transformation through rotation

We invite the reader to examine A.3.12 and A.4.16.

2.5.1a: *A "glissando" sound* (2.1.2) *interpreted as the transformation through rotarion of a sound of constant pitch*

Let S be a sound of constant pitch and of Δ duration defined in sound space $\mathbf{R}^2$ by the points A and D of a, d abscissae (its appearance and disappearance moments). The transformation of S into a glissando sound S' may be interpreted as a transformation through rotation, the rotation center being located in the point A or in a point of abscissa t ($a < t < d$). (Depending on the composer's directions or on the interpreter's mode of performance, a glissando may unfold on the whole duration of a "note" or on a part of it only) (Fig. 2.82).

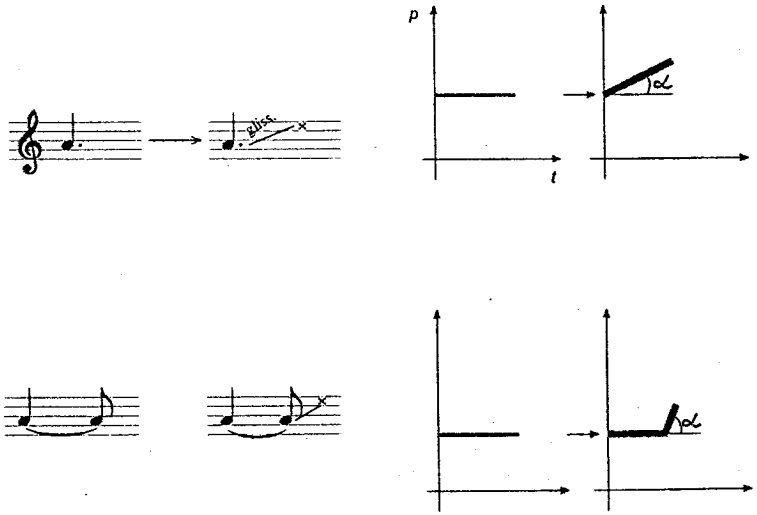


Fig. 2.82

Except the point A, representing the rotation center, the transformed image of S takes different positions in $\mathbf{R}^2$ depending on the rotation angle α. Pitch, duration, direction and sense (2.1.3) of sound S become a function of α value.

2.5.1b: *Transformation of a sound of constant pitch into a glissando sound of equal duration to the initial one: rotation composed with homothety*

As a rule, glissando is used as a connection between two concatenated sounds S and S_1 of different pitches, the amplitude Am representing the difference of the two sound ordinates. Usually, a glissando begins somewhere at a moment t during the unfolding of S (2.1.2) (Fig. 2.83a, b, c).

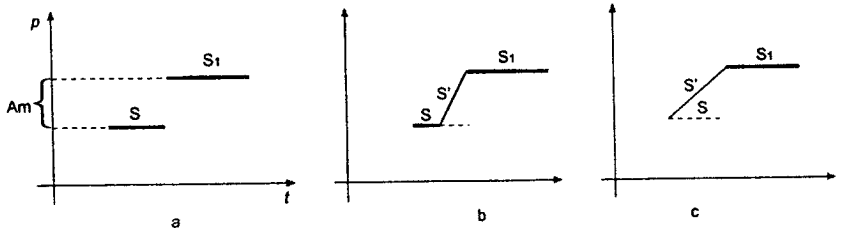


Fig. 2.83a Fig. 2.83b Fig. 2.83c

In the last case (2.83c) the duration (Δ) of the glissando sound is equal to that of the initial sound (S): $\Delta(S) = \Delta(S')$. The transformation appears as a rotation composed with a homothety (Fig. 2.84).

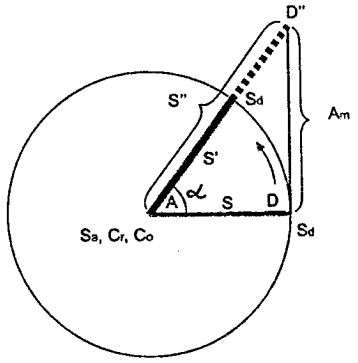


Fig. 2.84

Indeed the sound S" defined by the points A, D" may be interpreted as being obtained from S through:

(1) rotation (2.5.1c) of angle α, from a center C_r situated in A.

$$\alpha = \text{arc tg } \frac{Am}{S}$$

composed with

(2) homothety (A.3.15) of center C_0 also situated in A, of ratio

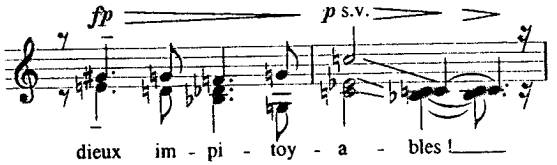
$$q = \frac{\sqrt{Am^2 + AD^2}}{AD}$$

$(S' = S : AD = AD')$

(1) transforms S in S' and (2) transforms S' in S''. The angle α represents *the slope* of the glissando.

2.5.1c: *The rending of a glissando through rotation and homothety allows a more accurate notation of the mode. Such a type of musical sound is performed depending on time parameter (t)* (2.1.2)

Let us take for example a glissando in the Prologue to Enescu's *Œdipe* (Fig. 2.85).

Fig. 2.85

If the soprano glissando begins simultaneously with the appearance moment of musical sound C_4 ($t = 0$) its slope (α) will be $\alpha = -36.06380$ and the value of the homothety ratio k will be $q = 1.25$ (Fig. 2.86).

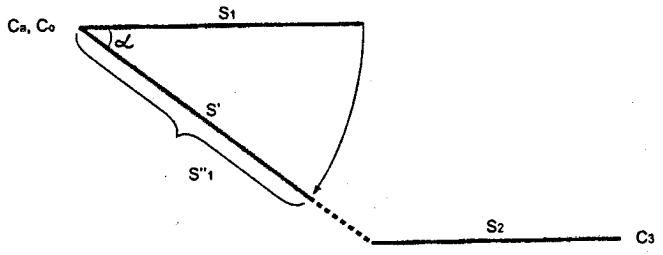


Fig. 2.86

If the glissando begins at $t = 9$, unfolding on the last seven sixteenth notes of the considered sound, its image will be (Fig. 2.87).

$(\alpha = -59.7435..., k = 1.98463)$

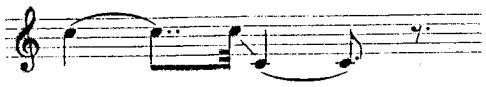
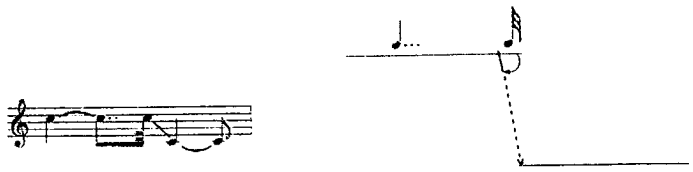


Fig. 2.87

Here below is the image for $t = 15$: $\alpha = -80.53$; $k = 6.08$. Glissando occurs on a thirty-second note time length (Fig. 2.88a).

Fig. 2.88a



It is easy to notice that α can take values only in the interval $0° < \alpha < 90°$. In order to have $\alpha = 0°$, the two sounds should have the same pitch, hence a glissando would be impossible. For $\alpha = 90°$, the glissando would become a cluster (2.1.2) of null duration, Fig. 2.88b.

Fig. 2.88b

2.6. TRANSFORMATIONS IN VIRTUAL SOUND SPACE R^3

We will recall some concepts:

1. The sound form is a connected set of null and non-null coplanar sound points.

2. The sound form space is the smallest rectangle including all the points of non-null sound events belonging to the given sound form.

3. The virtual sound space $\mathbf{R}^3$ is an operational space, two of its dimensions being those of the real sound space $\mathbf{R}^2$, the third one being an abstract algebraical dimension. Virtual sound forms are transformed in $\mathbf{R}^3$ and projected into the real sound space $\mathbf{R}^2$.

We suggest the reader re-examine paragraphs 1.1.1–1.1.5; A.3.15–A.3.20.

A. Virtual space R^3 endowed with affine structure

2.6.1. Transformations through translation composed with projection

Transformation of a sound duration interpreted as a transformation through projection from the virtual sound space R^3 into the real sound space R^2. The value of a pitch interval may be expressed by the value of a projection angle.

Let S be a sound in the real sound space $\mathbf{R}^2$ (Fig. 2.89a), the points A and D representing its appearance and disappearance moments.

Recalling the cases in 2.3.2 and considering m and n to be the variables defining the transformations of the appearance and disappearance moments of a sound,

$$x(A) + m < x(S) < x(D) + n$$

the modification of the appearance date may be written down as

$$m \neq 0;\ n = 0.$$

For $m > 0$ the appearance moment will be anticipated; for $m < 0$ it will be delayed.

Such a transformation may be interpreted as a transformation through translation of a sound S from $\mathbf{R}^2$ into $\mathbf{R}^3$: $S \rightarrow Sv$; $z(Sv) = h$,[11] composed with a projection of the virtual sound S' into the real sound space $\mathbf{R}^2$ from a projection center CP (Fig. 2.89b).

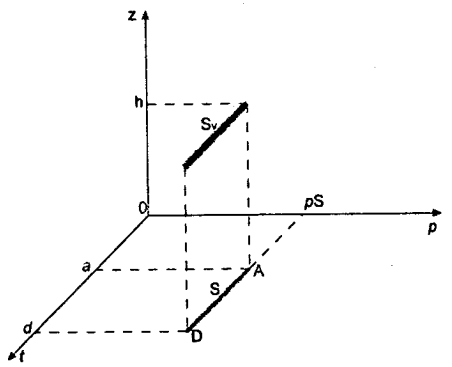


Fig. 2.89a

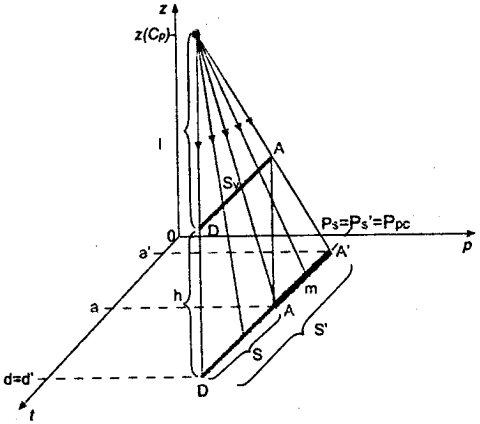


Fig. 2.89b

Cp is situated in a plane which includes S and is perpendicular to the real sound space, having as coordinates $x(CP) = x(D)$; $y(CP) = y(A,D)$; $z(CP) = h + l$; where $z(CP) > h$, and l is the distance from Sv to CP.

Since the segment Sv congruent with S represents the duration Δ of the considered sound, $\Delta = x(D) - x(A)$, the following relationships may be written down:

[11] The real sound S becomes the virtual sound S_v; by h is denoted the "height" where S has been "raised".

$$\frac{(h+l)}{l} = \frac{(\Delta+m)}{\Delta} \quad (1) \qquad m = \frac{(\Delta h)}{l} \quad (2)$$

(Thales)

Since h and Δ have constant values, it follows that $m = f(l)$, i.e. the m value depends on the distance from Sv to the projection center, a fact which is easily perceptible. For $l \to \infty$, $m \to 0$. Cp being situated at infinity, the projection becomes cylindrical; when the projections are orthogonal, the duration of Sv is equal to that of S: $\Delta S' = \Delta Sv$ (A.4.1) (Fig 2.90a). For $l \to 0$, $m \to \infty$, m tends to have infinite duration, hence S' starts somewhere in infinity (Fig. 2.90 b).

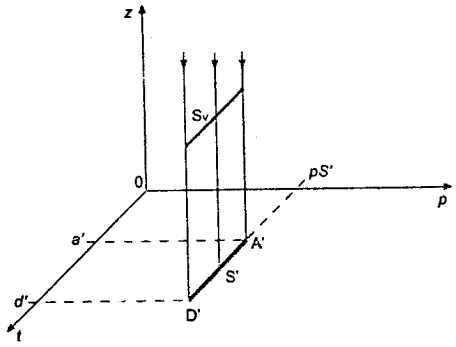


Fig. 2.90a

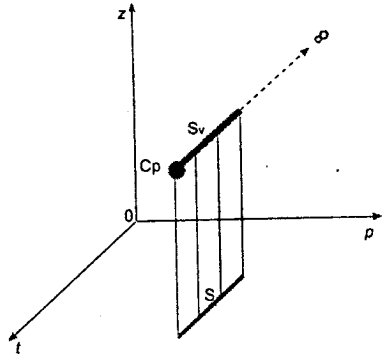


Fig. 2.90b

2.6.2. Numerical example

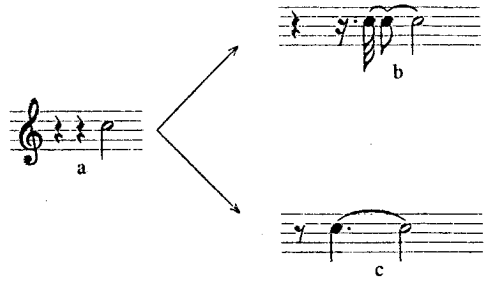
If we wish to obtain the transformation suggested in Fig. 2.89b, i.e.,that the musical sound C4 defined by the vector 16 12 32 12 (Fig. 2.91a) should anticipate its appearance moment by five thirty-second notes (Fig. 2.91b), m should have the value 5. Assuming S has been "raised" in $\mathbf{R}^3$ to the position $h = 12$, from the previous formula (2)

$$l = 12 \times 16 / 5 = 38.4$$

hence $z(Cp) = 50.4$.

If the sound S should start a dotted quarter note (three eigth notes) earlier (Fig. 2.91c), Sv should then be projected from a center situated at $z = 28$.

Fig. 2.91a,b,c



2.6.3. The delay of the appearance moment of a sound S (case $m < 0$, $n = 0$) interpreted as its projection from R^3 into R^2, from a projection center Cp of coordinates $x(Cp) = x(D)$; $y(Cp) = y(A,D)$; $z(Cp) = u$ $u < 0$ (Fig. 2.92)

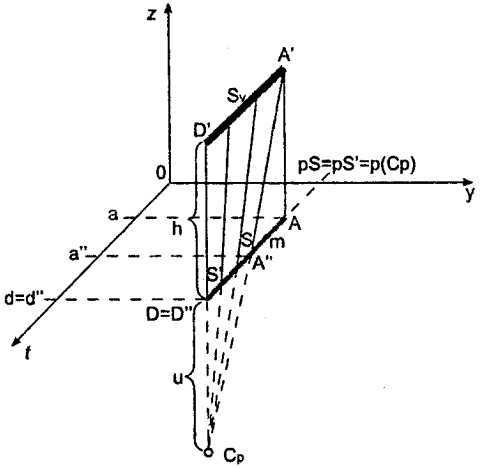


Fig. 2.92

The segment u is defined by the points Cp and D. Like in the case shown in (2.6.1a), $m = f(u)$. When $u \to 0 \Rightarrow -m \to S$, $S' \to 0$. The set pf points Sv agglomerates in point D' (the transformation biunivocity's character is jeopardized).

When $u \to -\infty$, $-m \to 0$ (2.6.1a) the projection lines become parallel.

2.6.4. Modification of the disappearance moment of a sound expressed as function of parameters m, n, h and $l(u)$

In the sense of those mentioned in 2.6.1, the modification of the disappearance moment of a sound may be expressed as $m = 0$, $n \neq 0$: for

$0 < n$, the disappearance moment will be delayed (Fig. 2.93); for $n < 0$, the disappearance moment will be anticipated (Fig. 2.94). For $0 < n$ the projection center will have the coordinates $x(Cp) = x(A)$; $y(Cp) = y(A,D)$; $z(Cp) = h + u$. For $n < 0$, $z(Cp) = u$, $z(Cp) < 0$. It is easy to remark that, like in the cases shown in 2.6.3 $(m = f(u))$, $(n = f(u))$.

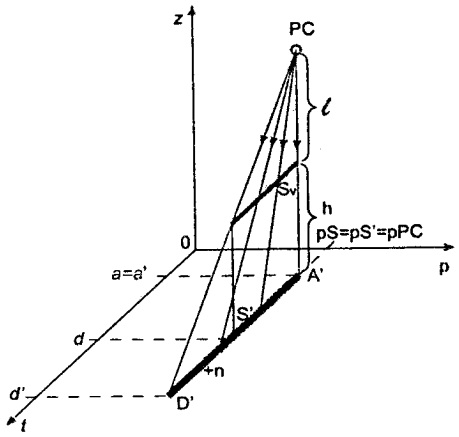


Fig. 2.93

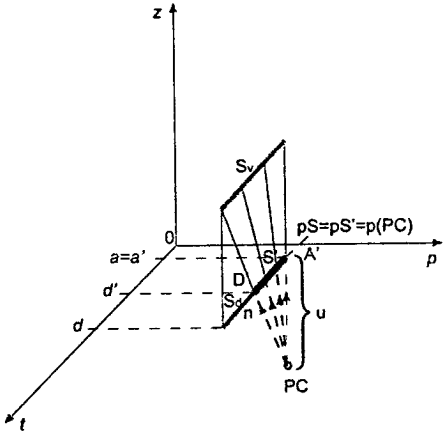


Fig. 2.94

The simultaneous modifications of appearance and disappearance moments may be interpreted in the same way (Figs. 2.95, 2.96, 2.97).

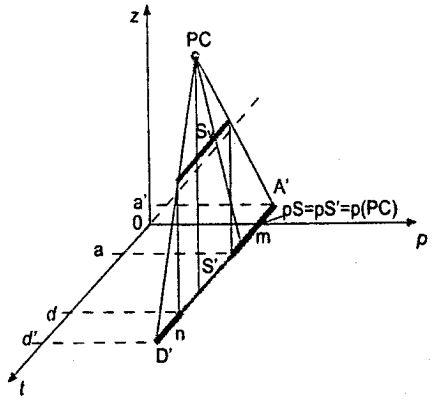


Fig. 2.95

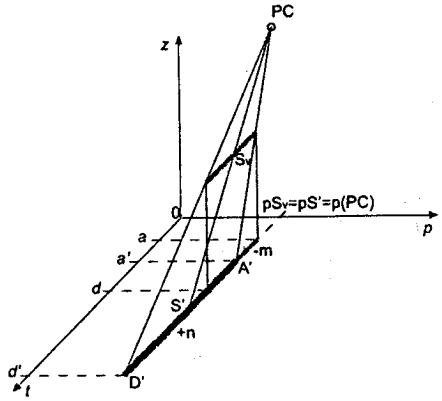


Fig. 2.96

In order to ilustrate by numerical examples the above shown transformations, let S be the musical sound B_3 (Fig. 2.98), defined in $\mathbf{R}^2$

Fig. 2.97

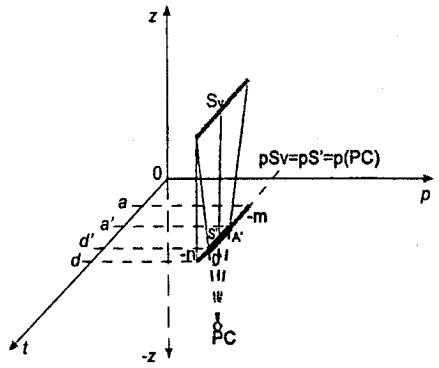
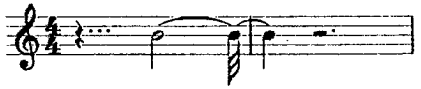


Fig. 2.98

by the vector 15, 11, 40, 11 (2.1.2).

Supose S has been "raised" in the virtual sound space $\mathbf{R}^3$ at the "height" h = 18. Considering that the projection center Cp is located successively in five different points of coordonates x, y, z = 15, 11, 56; 15, 11, –20; 20, 11, 56; 10, 11, 56; and 20, 11, –20, in musical notation the respective transformed images are Figs. 2.98a–e.

Fig. 2.98a

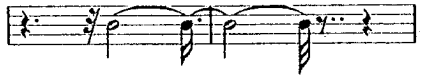



Fig. 2.98b

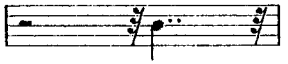



Fig. 2.98c

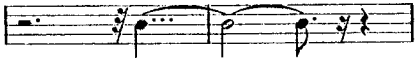



Fig. 2.98d

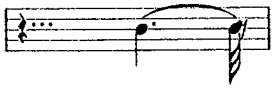



Fig. 2.98e

2.6.5. Sound pitch modification interpreted as a transformation by projection in the virtual sound space $\mathbf{R}^3$

Referring to the explanation in 2.3.2 and having q and r as variables defining the transformations of sound pitch at the A and D moments:

$y(A + q) < y(S) < (yD + r)$, the transformation of a sound of a given constant pitch $y(A) = y(D)$, $q = r$, into a sound of different constant pitch, may be interpreted as a transformation by *cylindrical oblique projection in* R^3. The sound pitch depends on the α angle of the oblique projections with the orthogonal ones when they intersect S_v.

The example in Fig. 2.99 represents an increased pitch of the transformed sound S' as against the pitch of the "source" sound S, this increase corresponding to the positive value of the α angle.

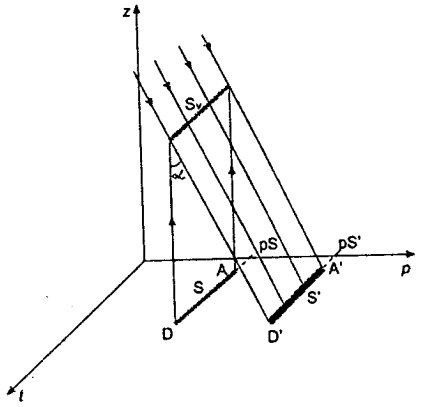


Fig. 2.99

The decrease of the pitch of S (the transformation $q < 0$, $r < 0$; $q = r$) corresponds to a negative value of the α angle (Fig. 2.100).

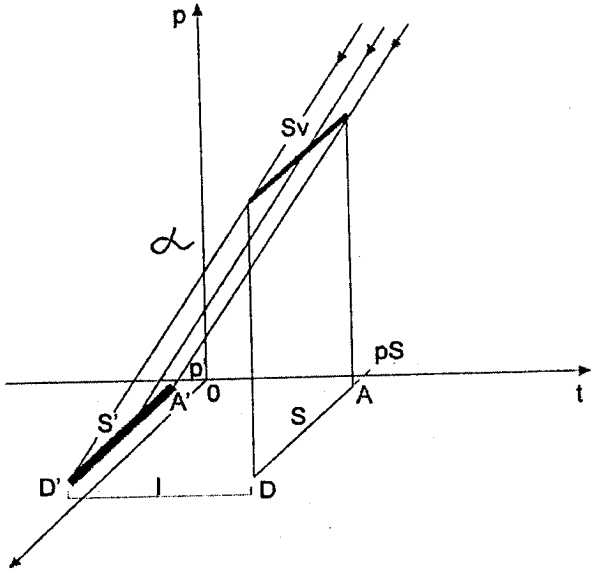


Fig. 2.100

2.6.6. The value of a pitch interval (I) may be expressed as $I = z(S_v) * tg\alpha$ where $z(S_v)$ is the "height" to which a sound S has been "raised" in the virtual sound space R^3, and α represents the angle of the orthogonal cylindrical projections with the oblique cylindrical projections when they intersect S_v

In order to build an interval I from a virtual sound Sv of coordinate $z = h$, one will make use of an angle $\alpha = \text{arc tg}\, I/h$ (Fig. 2.101a).

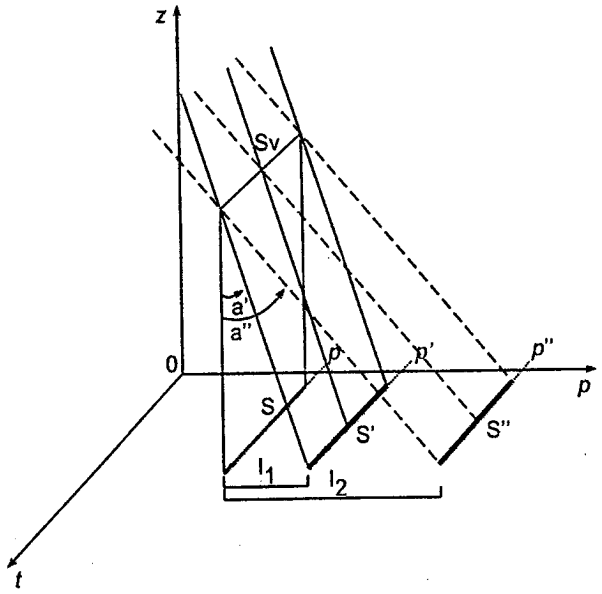


Fig. 2.101a

Assuming that the initial sound S has the pitch A_3 flat, in order to obtain out of this sound the pitch intervals of unison, ascending ninth, ascending major sixth + octave and descending fifth, the angle α has to take the values 0°, 37.87°, 48° and –18.45°, respectively (Fig. 2.101b).

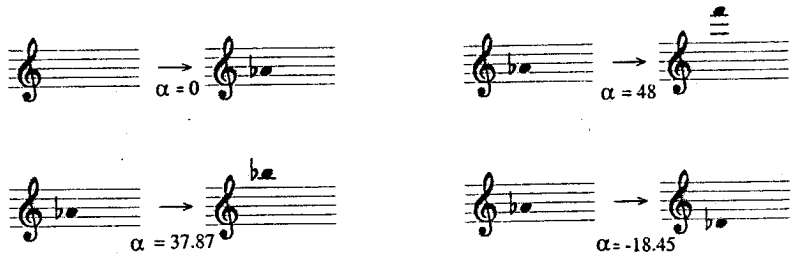



Fig. 2.101b

2.6.7. Simultaneous transformations of duration and pitch of a sound

A sound pitch increase and decrease joined to a modification of the appearance or disappearance moments may be interpreted as being obtained by the projection of *Sv* from the virtual sound space into the real sound space from a center *Cp* situated at finite distance.

If $x(A) < x(Cp) < x(D)$; $y(Cp) > y(S)$; $z(Cp) > z(Sv)$, then $y(S') < y(S)$; in other words the transformed sound S' will have a lower pitch than the initial sound S ($q < 0$, $r < 0$; $q = r$) and the values $m > 0$, $n > 0$.

If $x(A) < x(Cp) < x(D)$; $y(Cp) < y(S)$; $z(Cp) > z(Sv)$, then $y(S') > y(S)$: the transformed sound S' will have a higher pitch than the initial sound S, ($q > 0$, $r > 0$, $q = r$), and $m > 0$, $n > 0$ (Figs. 2.102a; 2.103a).

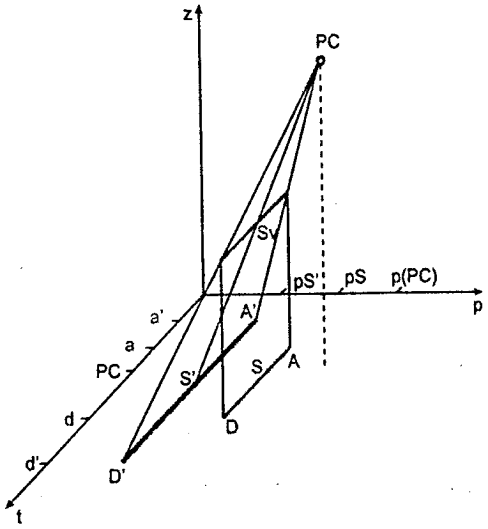


Fig. 2.102a

Fig. 2.102b

For example, referring again to the sound *S* previously defined (2.6.4; Fig. 2.98) and raised at h = 18 but projected now in the real sound space from the projection centers located *x*, *y*, *z* = 22, 17, 56 and 22, 7, 56, we obtain, in musical notation, the sounds shown in Figs. 2.102b and 2.103b.

It is easy to prove that for $z(Cp) < 0$, we will obtain besides a sound pitch modification, a modification of its appearance and disappearance moments also, in the sense of the diminution of the duration of *S*.

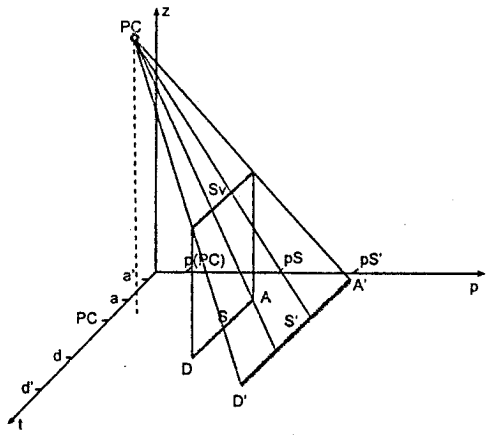


Fig. 2.103a

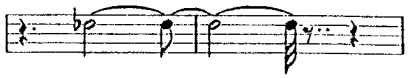



Fig. 2.103b

2.6.8. A musical chord may be expressed by a family of projection angles

As mentioned in 2.4.3, a musical chord is a family of musical intervals of common origin. Since a musical interval may be expressed by a projection angle, it follows that a chord may be expressed by a family of such angles (Fig. 2.101a).

B. Virtual sound space R^3 endowed with Euclidean structure

2.6.9. Transformations by translation composed with rotation and projection

Some homeomorphic images of the type of those obtained by translating a sound from $\mathbf{R}^2$ into $\mathbf{R}^3$ and by projection back into $\mathbf{R}^2$ (2.6.1 ... 4) may be also obtained by transforming the positions of points *A* and *D* (2.6.1) with respect to the projection center. In the following example, we may remark how the same transformation regarding the diminution of a sound duration may be obtained by projection from a center $z(Cp) < 0$, as well as by a rotation composed with an orthogonal cylindrical projection (Figs. 2.104, 2.105a,b).

Fig. 2.104

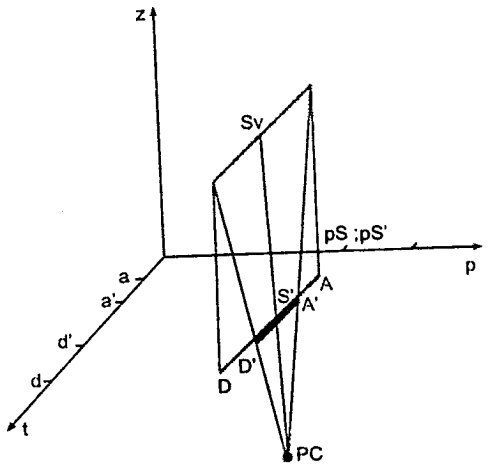


Fig. 2.105b

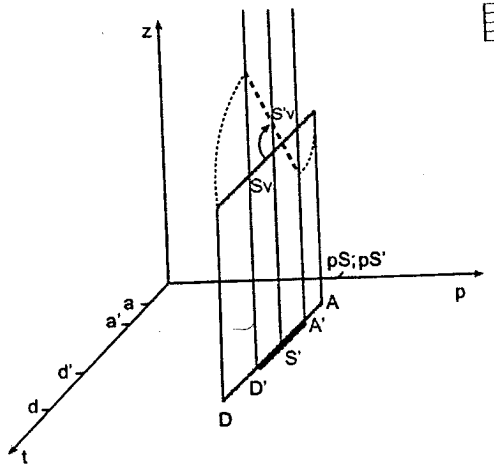


Fig. 2.105a

The duration of the transformed sound S' is variable in terms of the angle α made by S_v with S_v', particularly when:

(1) $\alpha = 0o \Rightarrow S' = S$ identical transformation (the S' duration is equal to that of S);

(2) $\alpha = 90o \Rightarrow S'$ becomes null sound (zero duration).

2.6.10. Group T (2.4.8) interpreted as a subgroup of the group of Euclidean space transformations

In the following we will include the group T as a subgroup of the group of Euclidean space transformations, i.e., as a projection-rotation composed

transformation in $\mathbf{R}^3$. As the musical term "reversion" ("renversement" in French and "Umkehrung" in German) shows, the denomination of this method of musical composition (1.5.7) implicitly suggests the idea of rotation.

Let F be a virtual sound form situated in a plane parallel to the real sound space $\mathbf{R}^2$. F represents the real sound form in $\mathbf{R}^2$, defined by the matrix (2.1.3)

x	y	x	y
0	16	8	16
8	11	20	11
20	7	28	7

and "lifted" by translation with respect to a perpendicular vector on $\mathbf{R}^2$ to a certain "height" z. Let z be 25. Since virtual sounds are defined by the coordinates of points defining their appearance and disappearance moments (x) and by the y and z values corresponding to these moments, F is defined in $\mathbf{R}^3$ by the following matrix (2.1.3):

x	y	z	x	y	z
0	16	25	8	16	25
8	11	25	20	11	25
20	7	25	28	7	25

For an easier visualization of a rotation in $\mathbf{R}^3$ we will show in the transformational process that part only of the plane corresponding to the rectangle which represents the sound form space (1.1.4).

By projecting F through orthogonal cylindrical projections into the real sound space $\mathbf{R}^2$, we will obtain the real sound form F_0, i.e., the *identical transformation* T_0 (Figs. 2.106a,b).

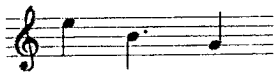



Fig. 2.106b

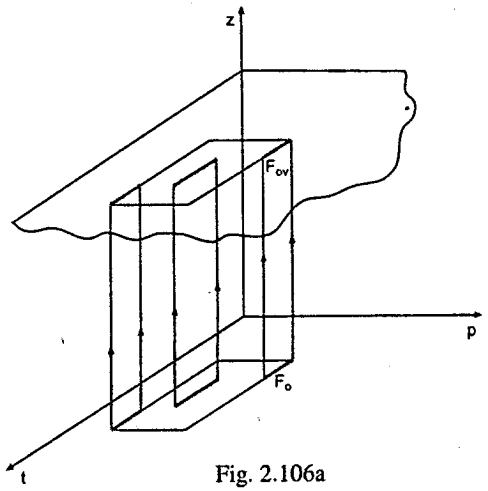


Fig. 2.106a

By subjecting F to a 180° rotation around an axis situated in the same plane as F, parallel to the time axis and crossing the minimum ordinate point of F, we obtain the homeomorphic image F'. Projecting F' into $\mathbf{R}^2$ we obtain the image F_1, i.e. the transformation T_1, the musical "reversion" (1.5.7) (Figs. 2.107a,b).

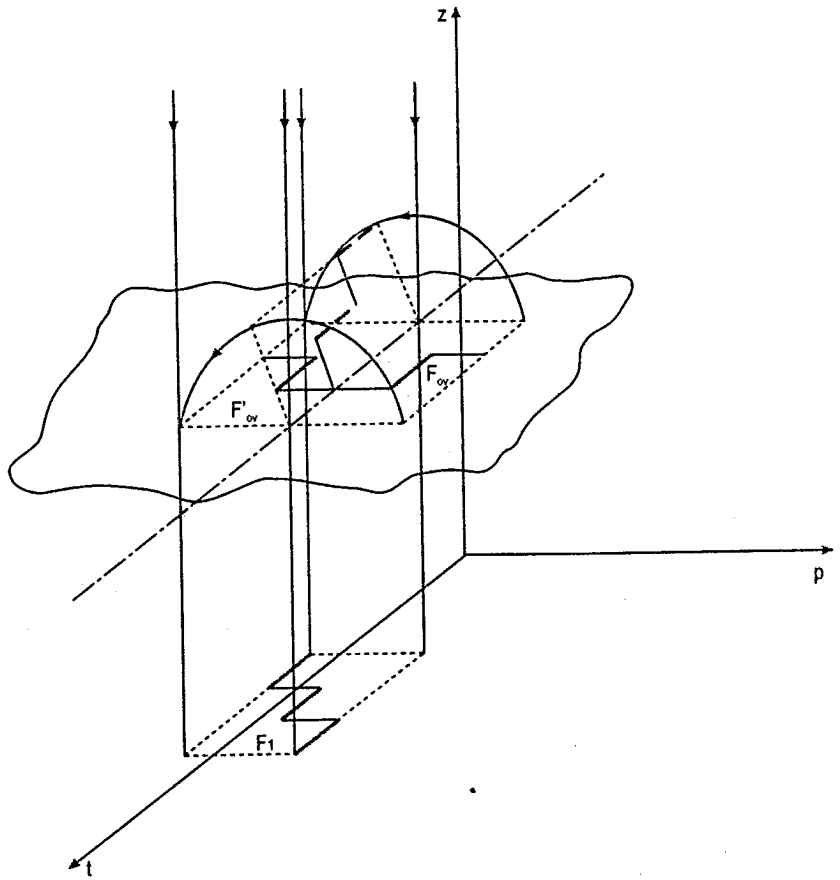


Fig. 2.107a

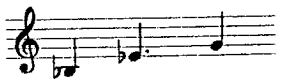



Fig. 2.107b

By subjecting F to a 180° rotation around a perpendicular axis on the time axis, which crosses the minimum abscissa point of F, we obtain the

homeomorphic image F'' which, by projection into $\mathbf{R}^2$ yields the image F_2, i.e., the T_2 transformation, the musical "inversion" (Figs. 2.108a,b).

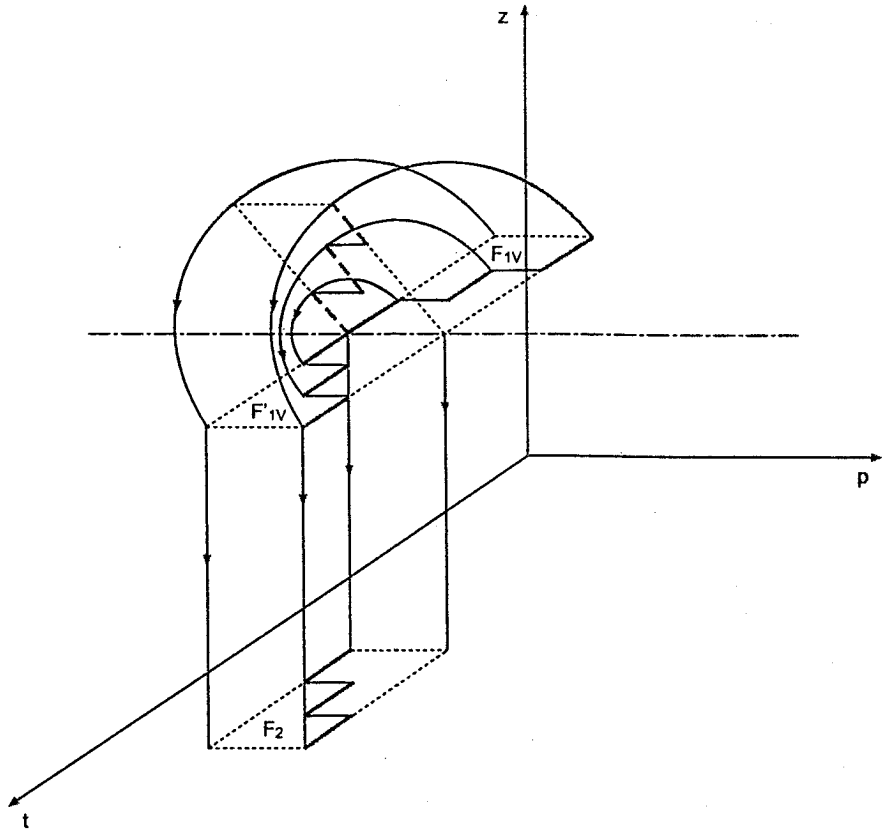


Fig. 2.108a

Fig. 2.108b

Finally, by rotating F'' around an axis parallel to the time axis and crossing the minimum ordinate point of F'', we obtain F''' which, when cylindrically and orthogonally projected into $\mathbf{R}^2$, will yield the image F_3, i.e., the transformation T_3, the musical "reversion of inversion" (Figs. 2.109a,b).

It may be easily proven that, by composing the images F_i, $i = 0, 1, 2, 3$, we obtain the table in 2.4.7 and, consequently, the commutative group structure of the four considered musical transformations is verified.

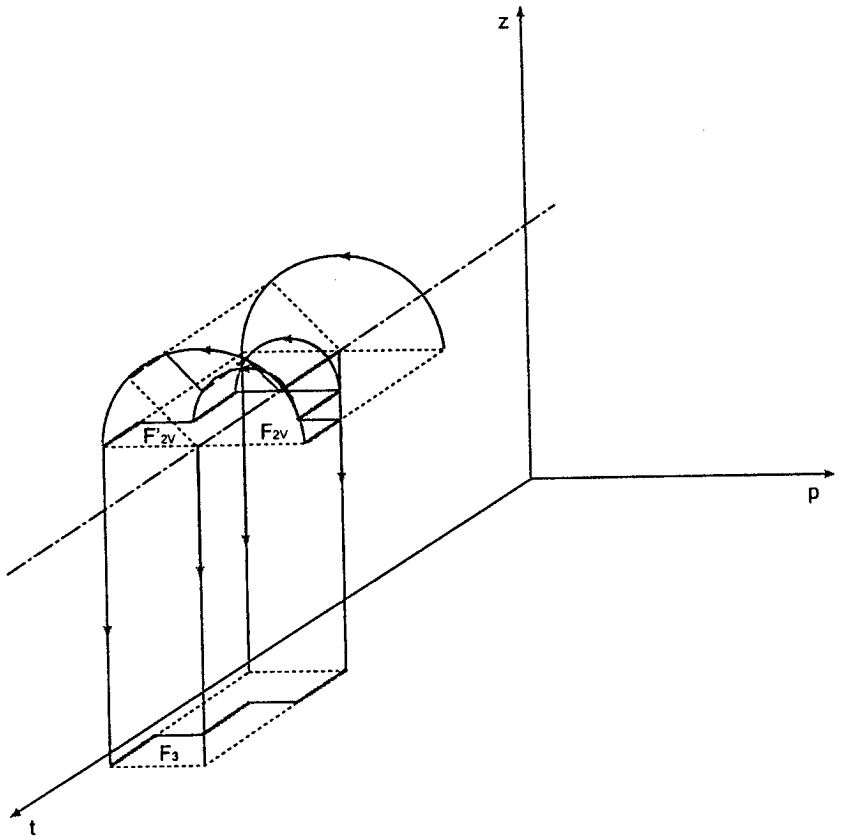


Fig. 2.109a

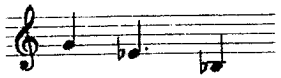



Fig. 2.109b

3. BEHAVIOUR OF SOUND FORMS UNDER THE GENERALIZED ACTION OF TOPOLOGICAL TRANSFORMATIONS WHICH EXHIBIT PARTICULAR CASES IN MUSICAL COMPOSITION METHODS. A POSSIBLE EXTENSION OF THESE METHODS

3.1. THE COMPOSER AND THE TOPOLOGICAL TRANSFOMATIONS

3.1.1. *Some conclusions regarding the preceding chapters*

In the second part of this work we have tried to prove that many musical concepts represented in the sound spaces $\mathbf{R}^2$ and $\mathbf{R}^3$ may bear a topological interpretation and that several classic composition methods may be viewed as particular cases of sound form transformations in topological spaces.

A question naturally arises: what is the behaviour of musical sound forms when they are subjected, in a general manner, to the action of various groups of topological tranformations? Do only the cases illustrated in the preceeding chapters have a musical significance? What kind of images are obtained when a musical sound form undergoes the same transformations as those used in traditional music, but other values are given to their characteristic parameters? What happens to a musical theme represented in real sound space $\mathbf{R}^2$ when transformed through homology, but, unlike the method leading to the classical inversions and reversions, the homology center is not situated at infinity, but at finite distances, the homology axis is not orthogonal on one of the coordinate axes and the homology ratio q is different from -1? Which is the image of a musical interval or chord, subjected to such transformations? What will happen when a sound form representing a musical fragment is transformed through homothety, and the

value of the homothety ratio is different from that leading to the old but always young "reversion of the inversion"? How is the image of a musical theme when, represented in virtual sound space $\mathbf{R}^3$, it is subjected to rotation around other axes and at other angles than those generating the classical group T (2.4.8)? Moreover, what happens to the sound form when the projection from the virtual sound space $\mathbf{R}^3$ into the real sound space is no longer cylindrical and orthogonal, but conic and obtained from projection centers situated at finite distances?

At last, what is going to be the result when topological transformations (other than those where certain composition methods of traditional music can be evidenced as particular cases), are applied to musical sound forms, as, for instance, conform transformations or transformations of sound forms consisting of non-coplanar virtual sound events in sound space $\mathbf{R}^3$?

These are the ideas underlying the third part of this work and the problems which it attempts to answer.

3.1.2. Ways of notating the parameters of some transfomations and special signs for defining particular musical sounds

In general, the initial form is denoted by F; its transfomed image by F'. In F the sounds are denoted by ciphers indicating their order of appearance: 1, 2,... n; in F' the corresponding sounds are denoted by 1',2'....n'. As a result some transformations acting upon F, the appearance order in F' may change. For instance, in Fig. 3.18, by reversed inversion of F, the appearance order becomes 3', 4', 2', 1'. The appearance order is indicated by ORD.

The homology center is denoted by C; the homothety, rotation, and projection centers by C_o, C_r and C_p, respectively. The homology axis bears the letter A and is represented by line segments and dots; homology and homothety ratios by q and k, respectively.

Some transformations may induce the change of direction of a vector representing a sound (A.3.16; 2.1.2; 2. 5. 1). In musical notation, the image of an "inversed sound" will be represented by an arrow ← placed over the symbol of the respective sound (Fig. 3.1a).

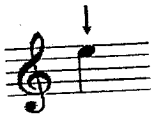



Fig. 3.1a

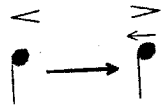


Fig. 3.1b

If some sounds are provided with volume indications, i.e. with "crescendo" or "diminuendo", the respective transformation may inverse the direction of the dynamics of the considered sounds: a crescendo sound turns to a diminuendo one and conversely (Fig. 3.1b).

As mentioned (2.1.3), a musical sound form is represented in sound space $\mathbf{R}^2$ by a matrix of four columns and n lines, each line representing a sound by means of the coordinates of the points defining its appearance and disappearance moments. As a rule, the coordinates are expressed as integer numbers, representing values in thirty-second notes (abscissae axis) and semitones (ordinate axis).

However, the results obtained through transforming operations frequently cannot be expressed through integer numbers. This being the case, the sounds defined by ordinates including decimal fractions of value smaller than 0.50, are represented by the symbol of the closest inferior sound, provided with an ascending arrow (↑); if the decimal part of the ordinate is equal or larger than 0.50, the respective sound is represented by the symbol of the closest superior sound provided with an descending arrow (↓). Example: the sound of ordinate 15.35 is marked with ↑ (Fig. 3.1c); the sound of ordinate 15.74 is marked with ↓ (Fig. 3.1d).

Fig. 3.1c

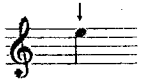



Fig. 3.1d

In order to have more clear-cut images, we have not introduced musical duration symbols of smaller values than the thirty-second note (1/64, 1/128).

If the duration of a sound or rest exceeds the value indicated by a musical duration symbol and the exceeding duration is less than 1/32 note (but not necessarily parts of this duration obtained by dividing it by 2, 2^2, 2^3 ...), the symbol is followed by the sign ">". If the duration of a sound or rest is smaller than the value indicated by a musical notation symbol, the symbol is followed by the sign "<".

The example in Fig. 3.1e shows that the sixteenth rest has a longer duration than two (1/32) units, and the duration of the eighth Eb_4 note is smaller than four such units. The sound Eb_4 is "compressed" from both sides.

Fig. 3.1e

However, again, in order to have images more clearly cut, but also because, anyway, when performing music small differences in duration

values are not noticeable (or can be viewed as topological transformations in the act of "interpretation" of a score (1.3.7)), in the musical fragments obtained through the various topological transformations we will be working with, often we will not make use of the signs ">" and "<".

Very often a certain transformation may change a sound of constant pitch into a "glissando" type sound (2.1.2). i.e., over a time length indicated by a musical duration symbol, the transformed sound covers a certain pitch interval. The mode of writing down such a sound is shown in Fig 3.2b: during the Eb_4 dotted quarter note, a glissando sound covers the pitch interval $Eb_4 - D_3$.

3.1.3. Compound transformations useful to the composer in the process of structuring the musical discourse. Ordinates lining up (OL) and abscissae ordering (AO)

If a sound form has several elements, situations may occur through its transformation where some special compound transformations could be useful to the composer.

Let us consider, for instance, the following musical fragment, the sound form *SF* (Fig. 3.2a) defined by the matrix

X	Y	X	Y
8	10	18	10
22	4	32	4
36	–6	52	–6

represented in sound space $\mathbf{R}^2$ as in Fig. 3.2a.

Fig. 3.2b

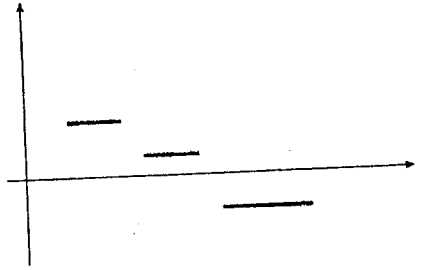


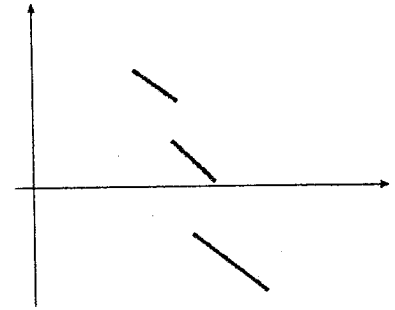
Fig. 3.2a

Anticipating the matters studied in the chapter regarding transformations through rotation (3.4.1), let us suppose that *SF* is transformed through rotation from the center of coordinates $x = 30$, $y = 0$ and an angle α, $\alpha = -40°$. SF_1, the image through transformation of *SF*, will be defined by the matrix:

X	Y	X	Y
19.57	21.80	27.23	15.37
26.44	8.20	34.10	1.77
30.73	–8.43	42.99	–18.76

appearing in the sound space $\mathbf{R}^2$ as in Fig. 3.3a,

Fig. 3.3a



and in a musical image as in Fig. 3.3b.

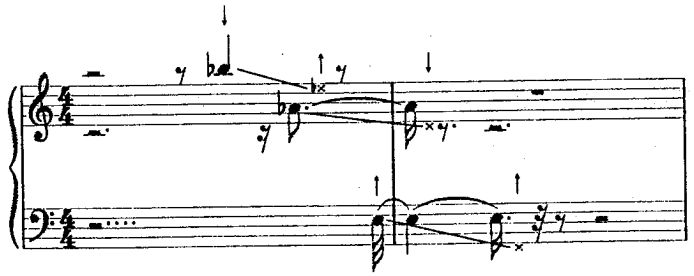


Fig. 3.3b

If the composer wants to avoid the "glissando" sounds, i.e. to line up the pitches of the sounds (the sounds should have a constant pitch), by another topological transformation (2.5.1) he could consider the form sounds as having the same duration as the *F* ones and as pitch, the pitch of the sound at its appearance moment. The new form *F"* will be defined by the following matrix:

X	Y	X	Y
19.57	21.80	27.23	21.80
26.44	8.20	34.10	8.20
30.73	–8.45	42.99	–8.45

having in sound space $\mathbf{R}^2$ the image shown in figure 3.4a

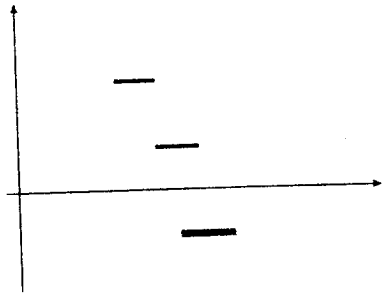


Fig. 3.4a

and in musical notation shown in Fig. 3.4b.

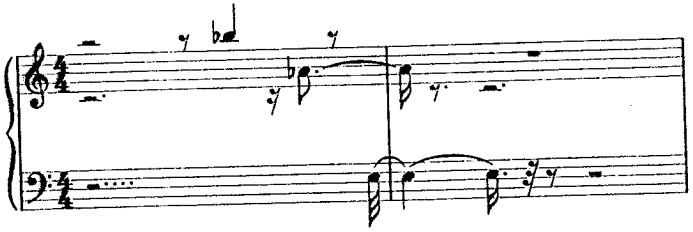



Fig. 3.4b

We will refer to this transformation as ORDINATE LINING UP, *OL*.

It should be noticed that both in *SF'* and in *SF"*, there are time intervals when the sounds overlap. If the composer wants to avoid this overlapping so as to have concatenating sounds, i.e. successive in time, he can make use of another transformation in that, when two or more sounds, whose projections on the time axis coincide on an interval, the duration of the sounds is reduced in the sense that, the dissappearance moment of a sound would coincide with the appearance moment of the following sound (overlapped sounds become concatenated sounds).

We will refer to this operation as ABSCISSAE ORDERING, *AO*.

Subjecting *F"* to a transformation through *AO*, the composer obtains *F"'*, defined by the matrix:

X	Y	X	Y
19.57	21.80	27.23	21.80
27.23	8.20	30.73	8.20
30.73	–8.45	42.99	–8.45

having in $\mathbf{R}^2$ the image (Fig. 3.5a)

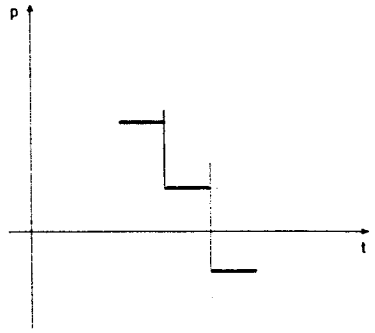


Fig. 3.5a

and in musical notation (Fig. 3.5b).

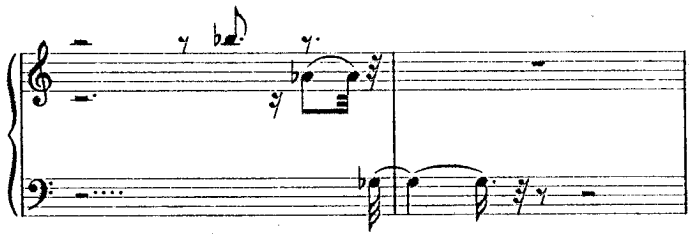



Fig. 3.5b

It is easy to see that *OL* and *AO* are topological transformations, more specifically homotheties (2.4.2), rotations and rotations composed with homotheties (2.5.1).

3.1.4. Using modal adjustment

Finally, in the transformation process of a set of sounds belonging to a certain mode or certain key, it is most likely that, by this transformation, all sounds or a strict part thereof should be situated at pitches which do not belong to the initial mode or key. If the composer desires the sounds to belong to the same mode or to another mode, he will be able to obtain this by a *modal adjustment* (*MA*; 2.3.9).

3.2. BEHAVIOUR OF SOUND FORMS UNDER THE ACTION OF HOMOLOGY TRANSFORMATION

We suggest you read again A.3.6 and A.3.8 concerning the transformations through homology.

3.2.1. Homology with its center at finite distance

In order to easily survey the behaviour of a musical sound form under the action of this transformation, depending on form positions with respect to the axis, center, and biratio q value, in other words in order to study from a musical point of view the transformations a certain set of musical sounds undergo in the three cases of homology with center at finite distance (Fig. 49a,b,c) as well as in the case of parallel homology (center situated at infinity (Fig. 51a,b,c), we will subject a sound form with a restricted number of sounds (liable to form a musical "theme") to the considered homology transformations. Let F be the sound form represented by the fragment in Fig. 3.6a

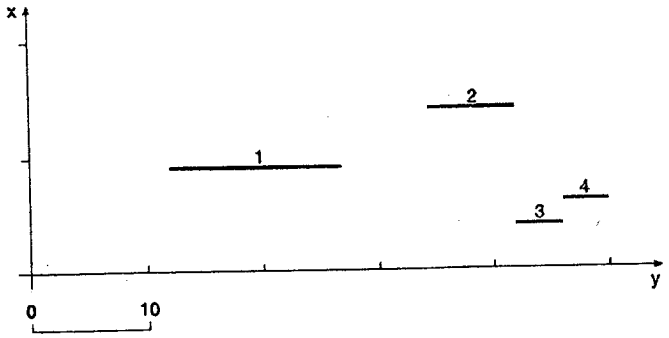
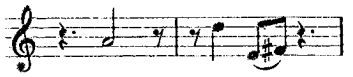


Fig. 3.6a

defined in sound space $\mathbf{R}^2$ by the matrix

x	y	x	y
12	9	28	9
36	14	44	14
44	4	48	4
48	6	52	6

and having in this space the image (Fig. 3.6b).

Fig. 3.6b

In the following, the critical values of biratio q viz. $\alpha / 1 - \alpha$, $\alpha / \alpha - 1$ and $\alpha / \alpha + 1$ are avoided for all points representing F in $\mathbf{R}^2$, viz. for all appearance (A) and disappearance (D) moments of the sounds (2.1.1).

According to the theorem mentioned at (A. 4.6), the biratio q values are outside the interval defined by the minimum and maximum values $\alpha / 1 - \alpha$ i.e. $\alpha / \alpha - 1$ and $\alpha / \alpha + 1$, respectively.

3.2.2. Case 1 (sound form situated between center and axis)

The homology center is defined by the point of coordinates (32, 20).

The homology axis is defined by the point of coordinates (32, 0) and the angle α made by the axis with the Ox axis, $\alpha = -10^\circ$.

The q values situated inside the (–2.65277, –0.241093) interval are avoided.

We remark:

For $q = 1$, $F = F'$ (identical transformation).

For $1 > q > 0$, F' is dilated, its component sounds are expanded, spread, "migrating" towards the homology axis.

For $q = 0$, the F' sounds "settle down" on the homology axis.

For $0 > q > -0.241093$, F further expands itself as $q \to -0.241093$. "Passing" over the homology axis, F reverses its image. For $-2.65277 < q < -0.241093$, F' has points to infinite, hence q cannot take values on this interval.

For $-2.65177 > q > -\infty$, F' appears from the opposite side of the axis (when we consider that F' has "migrated" to the "south", it appears from the "north"). F' is the "reversed inversed" image of F. F' shrinks, sounds tend to agglomerate in the homology center, as $q \to -\infty$.

For $q = \pm\infty$, all sounds of F' are agglomerated in the homology center. "Passing" through the homology center ($q = +\infty \to q = -\infty$) F inverses and reverses its image.

For $+\infty > q > 1$, F' expands, tending to take the form of F as $q \to 1$.

Expansions and contractions of musical sound forms through homology transformations may induce overlapping in time and modifications of sound

appearance order (3.2.1). Thus, in the example in Fig. 3.8, the sounds 3', 4' overlap 2'. Through the reversion of inversion of F, the sound order implicitly modifies. Also, when sounds are represented through vectors in $\mathbf{R}^2$, the reversion of inversion includes the sense inversion of the respective vectors (A.3.8. – A. 3.9a). This being the case, a sound represented by a segment AD is transformed in a segment $D'A'$, hence, a "decrescendo" sound in F' is obtained from a "crescendo" sound in F, and also inversely. Finally, except the case when the homology axis is parallel to the time axis, sounds of constant pitch (represented by segments parallel to the time axis) are transformed into "glissando" sounds (2.1.2). The glissando slope depends on the value of the angle made by the homology axis with the time axis.

The following figures represent the sound form and its transformations through homology in terms of the different values of the q parameter. To facilitate the study the behaviour of F under homology action, the values of the variables defining the positions of the homology axis and center are identical in all examples.

$q = 1$

Identical transformation (Fig. 3.7a, b)

12	9	28	9
36	14	44	14
44	4	48	4
48	6	52	6

ORD 1 2 3 4

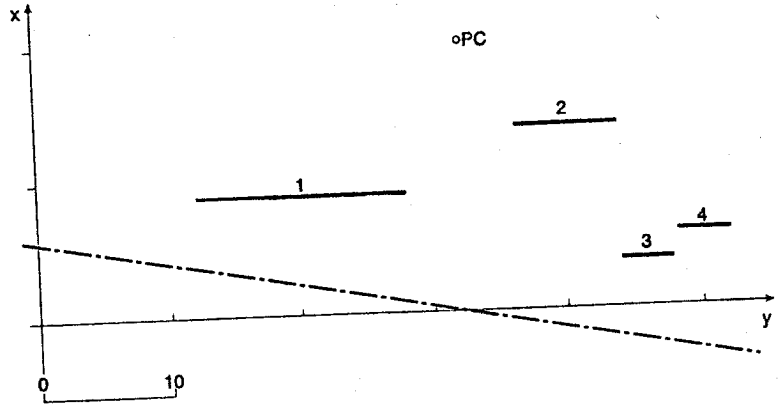
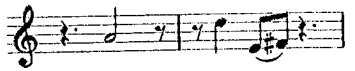


Fig. 3.7a

Fig. 3.7b

$q = 0.25$

$0 < q < 1$: F expands migrating towards the axis (Fig. 3.8a, b)

6.83	6.15	26.19	4.03
40.91	6.62	62.32	4.83
47.57	–0.76	53.49	–1.49
55.90	–0.91	63.11	–1.77

ORD 1 2 3 4

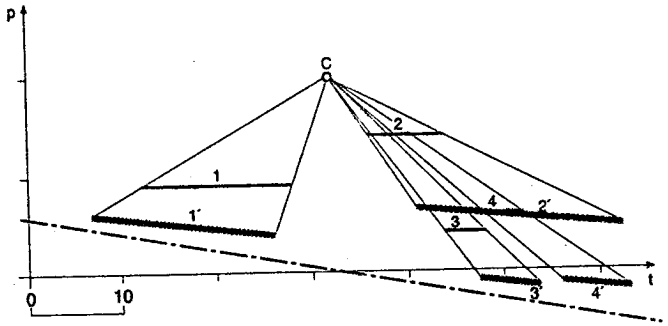


Fig. 3.8a

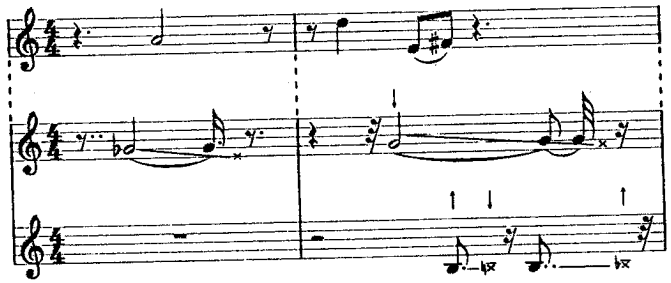



Fig. 3.8b

$q = 0$

F sounds "settle down" on the axis (Fig. 3.9a, b)

4.46	4.85	25.16	1.20
47.10	–2.66	93.77	–10.88
49.28	–3.04	56.27	–4.27
60.62	–5.04	70.18	–6.72

ORD 1 3 4 2

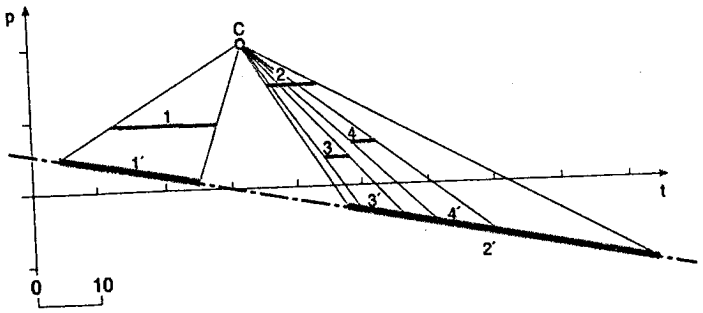


Fig. 3.9a

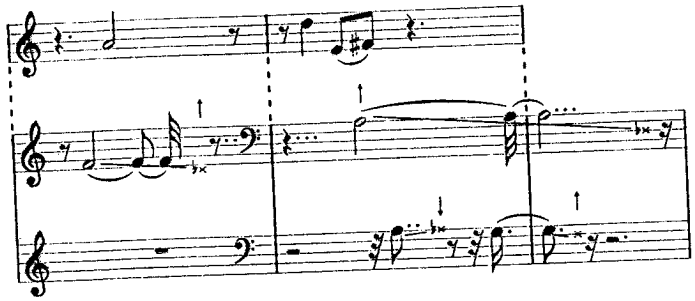



Fig. 3.9b

N.B. The topological aspect deteriorates: the transformation is no longer bicontinuous and biunivocal.

$q = -0.05$

$0 > q > -0.241093$ (max $\pi / \pi - 1$): *F* reverses in relation to the axis and extends. The 3' and 4' sounds overlap sound 2' (Fig. 3.10a, b)

3.98	4.36	24.91	0.51
49.54	–6.31	109.93	–18.96
49.67	–3.56	56.92	–4.92
61.79	–6.07	72	–8

ORD 1 2
3 4

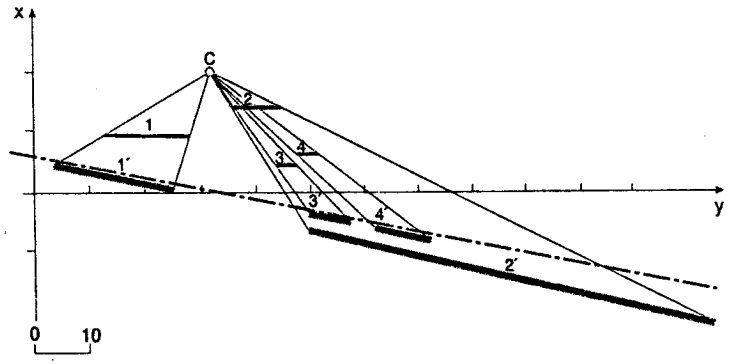


Fig. 3.10a

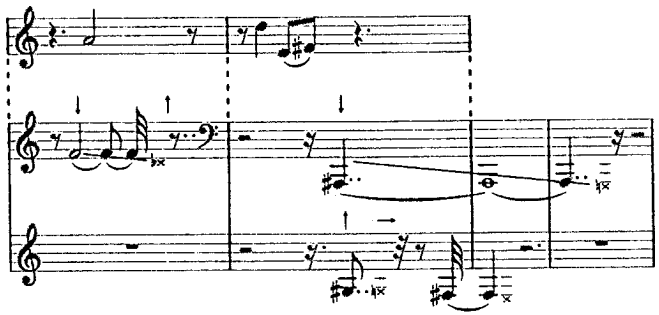


Fig. 3.10b

$q = -4.25$

-2.65277 (min $\alpha / \alpha - 1$) $> q > -\infty$: The *F* sounds tend to agglomerate in the homology center as $q \rightarrow -\infty$ (Fig. 3.11a, b)

77.73	45.15	35.39	29.34
30.60	22.09	28.28	21.85
12.16	46.44	11.75	40.25
19.83	30.64	18.66	29.33

ORD 3 4 2 1

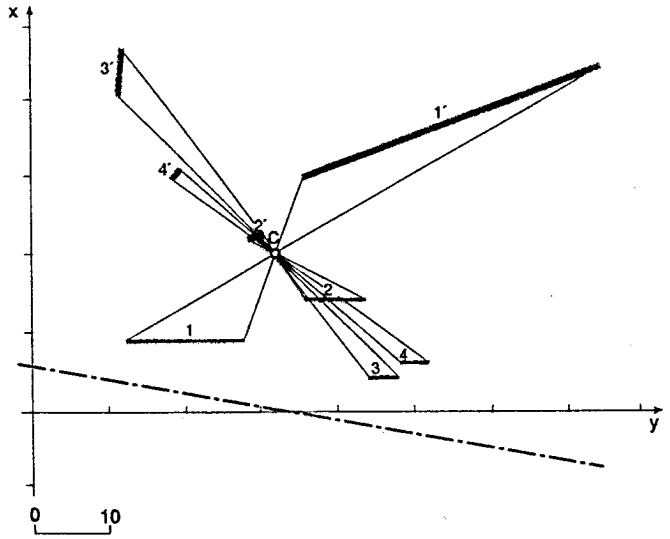


Fig. 3.11a

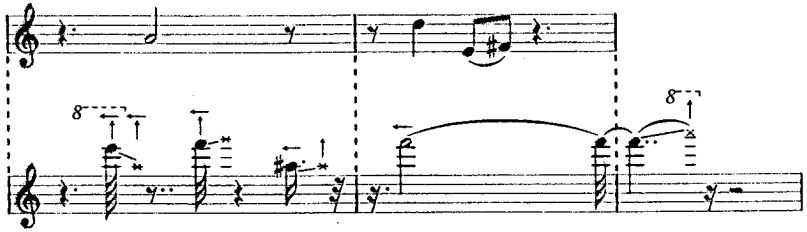


Fig. 3.11b

$q = 7$

$+\infty > q > 1$: F' draws near the F form as $q \to 1$ (Fig. 3.12a, b)

24.43	15.83	30.85	16.84
32.73	18.89	34.05	18.97
36.23	14.35	37.25	14.74
36.38	16.16	37.18	16.37

ORD 1 2 3
4

$q = 20$

$+\infty > q > 1$: F sounds tend to agglomerate in C as $q \to +\infty$ (Fig. 3.13a, b)

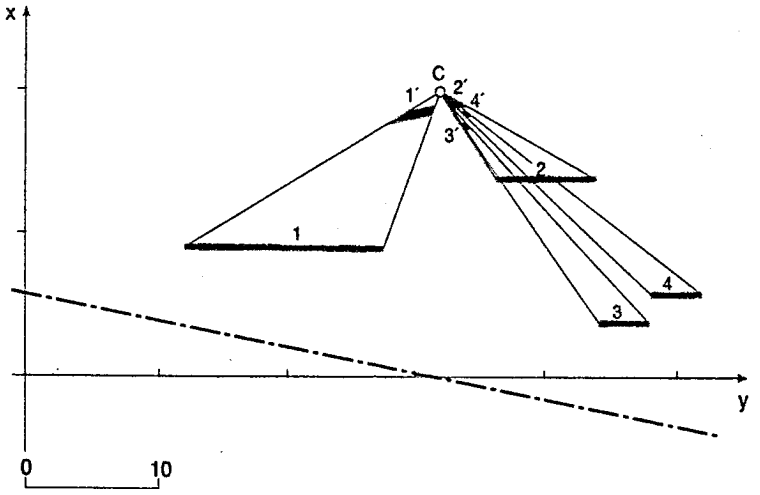


Fig. 3.12a

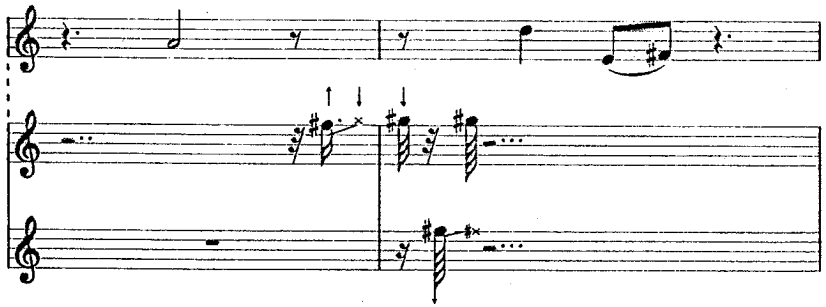



Fig. 3.12b

The reader is now quite familiarised with the homology transformation process, so that, if need be, by resorting to the relevant mathematical notions inserted in (A 3.6–8) the behaviour of a sound form under homology action could be surveyed in the other two cases. Here are some illustrative examples specific for each of these two cases.

28.77	18.22	31.54	18.76
32.26	19.59	32.73	19.63
33.76	17.65	34.13	17.86
33.7	18.5	33.99	18.6

ORD 1 2 4
3

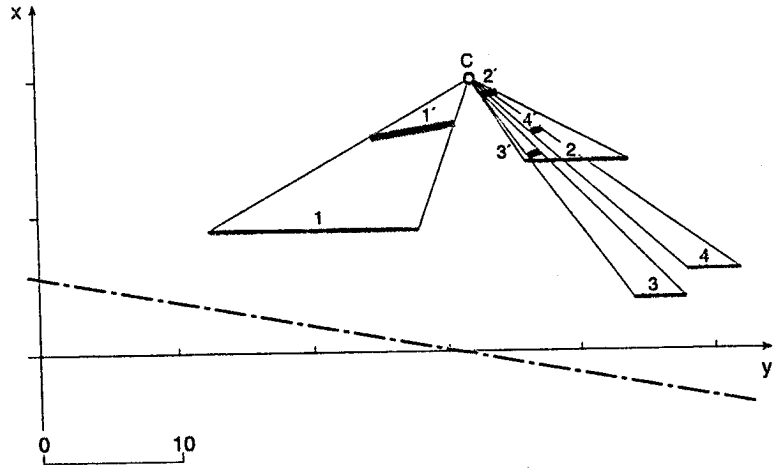


Fig. 3.13a

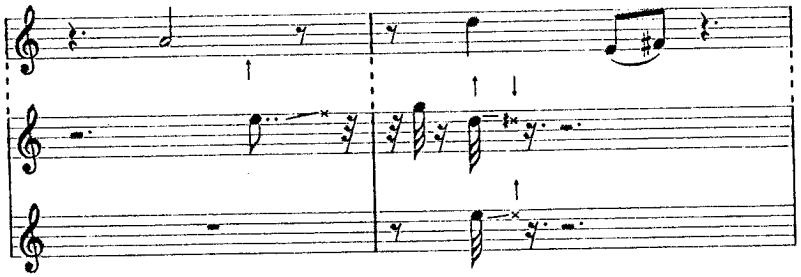



Fig. 3.13b

3.2.3. Case II (axis situated between center and sound form)

The center is situated at the point of coordinates (32, –16). As in the preceding case, the axis is defined by the point of the coordinates (32, 0) and the angle α, $\alpha = -10°$. The biratio q values within the (1.99281, 3.92222) interval, i.e. min π / 1 – π and max π / 1 – π, are avoided.

$q = 0.5$

O < q < 1: F' contracts and tends to "settle down" on the axis (Fig. 3.14a, b)

14.92	5.34	28.82	3.85
34.74	4.55	39.98	3.95
42.07	0.79	45.18	0.48
44.54	1.24	47.41	0.95

ORD 1 2 3
4

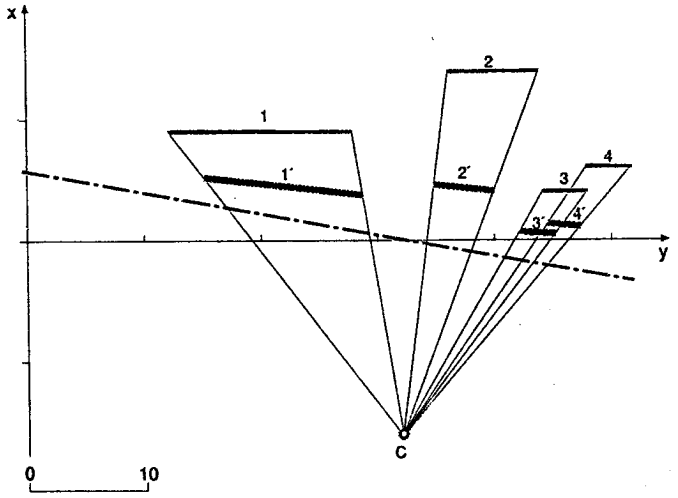


Fig. 3.14a

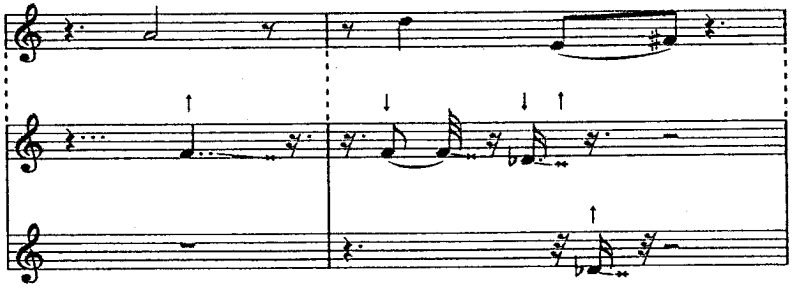



Fig. 3.14b

$q = 0$

F sounds are "settled down" on the axis (Fig. 3.15a, b)

17.89	2.6	29.36	0.46
34.08	–0.86	37.97	–4.05
40.66	–1.68	49.2	–1.97
42.31	–1.8	44.53	–2.20

ORD 1 2 3
4

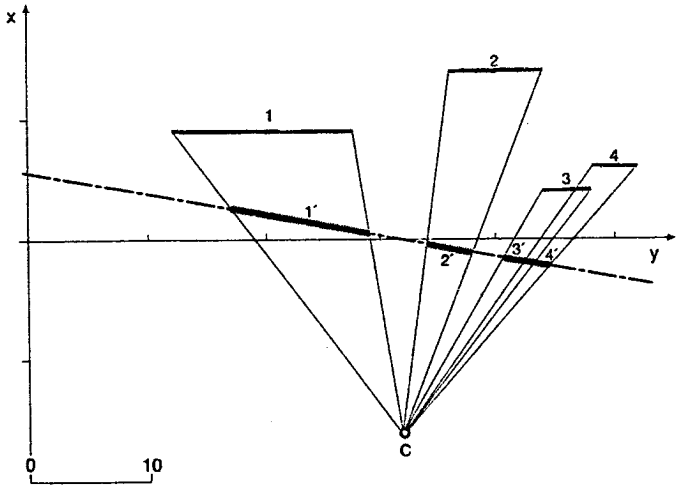


Fig. 3.15a

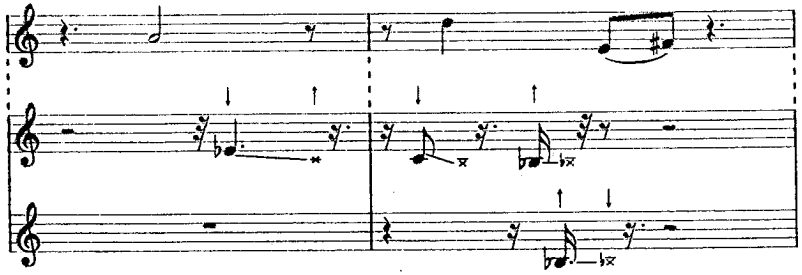



Fig. 3.15b

$q = -0.25$

$0 > q > -\infty$: F is reversed relatively to the axis, it further compresses and tends to the center (Fig. 3.16a, b)

17.92	1.5	29.57	–0.83
33.86	–2.03	37.31	–2.71
40.12	–2.46	42.43	–2.95
41.47	–2.97	43.46	–3.38

ORD 1 2 3
4

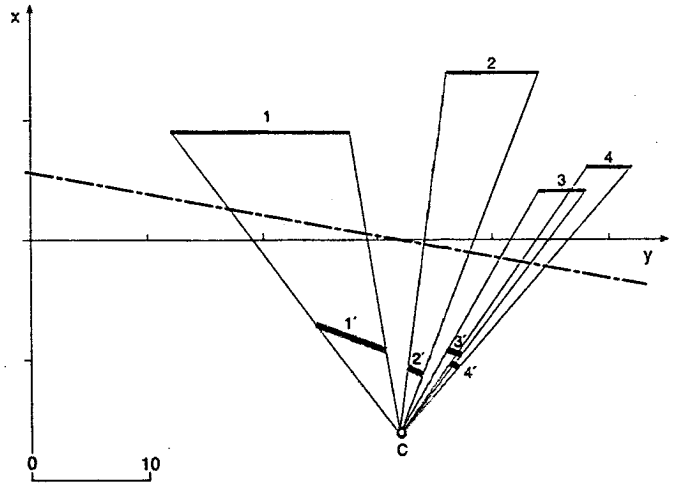


Fig. 3.16a

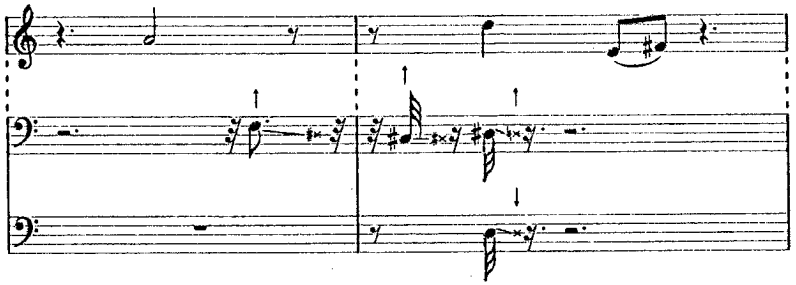



Fig. 3.16b

$q = -4$

$0 > q > -\infty$: F tends to agglomerate in the center as $q \to -\infty$ (Fig. 3.17a, b)

24.62	–6.77	20.88	–9.04
32.71	–10.63	33.98	–11.02
36.12	–9.12	37.10	–9.61
36.25	–10.14	37.02	–10.46

ORD 1 2 3
4

Fig. 3.17a

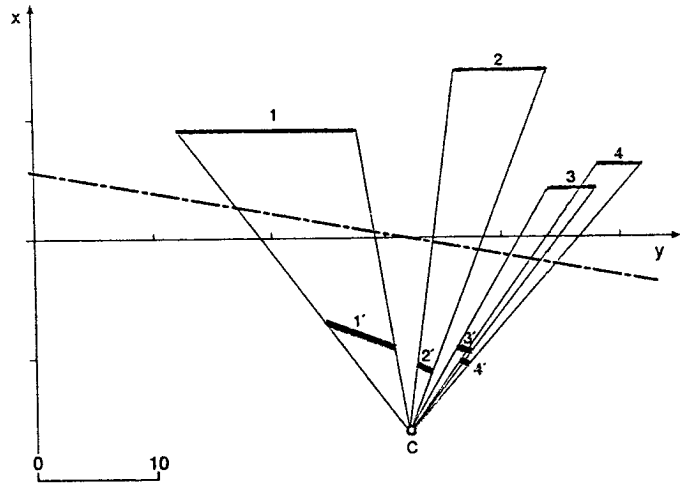
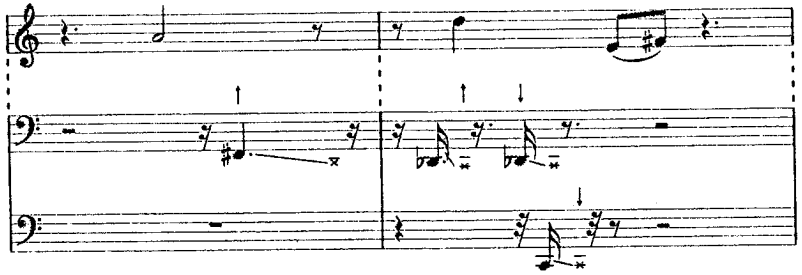


Fig. 3.17b

$q = 8$

$+\infty > q > 3.92222$ (max α / 1 – α): F' is reversed and inverted relatively to the center. F' extends and tends to "south" infinite (Fig. 3.18a, b)

46.38	–33.91	33.52	–25.50
31.26	–21.52	30.01	–20.95
24.83	–27.93	23.93	–26.08
26.40	–23.69	25.68	–22.94

ORD 3 4 2 1

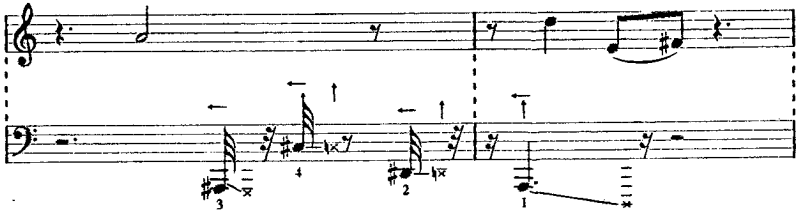
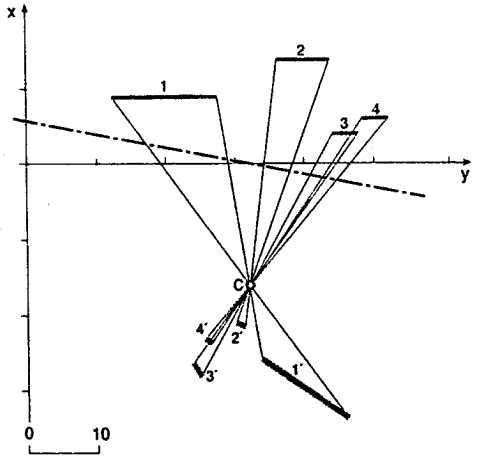


Fig. 3.18b

Fig. 3.18a



$q = 1.5$

$1.99281 > q > 1$: F "comes" from infinite "north" and tends to take the form F when $q = 1$ (Fig. 19a, b).

7.87	14.16	26.6	17.74
39.40	39.50	56.17	44.43
46.83	8.7	52.33	9.41
54.08	14.37	60.47	15.32

ORD 1 2

3 4

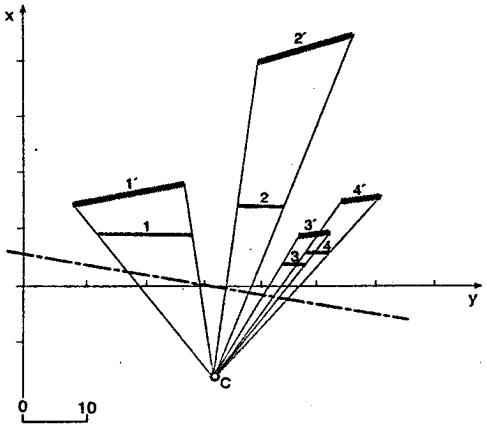


Fig. 3.19a

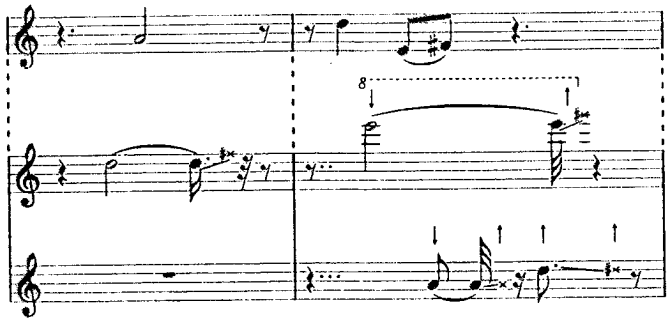



Fig. 3.19b

3.2.4. Case III (center between axis and form)

Center (32, –10); axis defined by the point of coordinates (32, –20); $\alpha = -10°$.

The values of biration q are avoided within the (0.607463, 0.723106) interval. (min. $\alpha / \alpha - 1$, max. $\alpha / \alpha + 1$)

$q = 0.85$

$1 > q > 0.723106$ (max $\alpha / \alpha + 1$): F extends and migrates "nonthwords" (Fig. 3.20a, b)

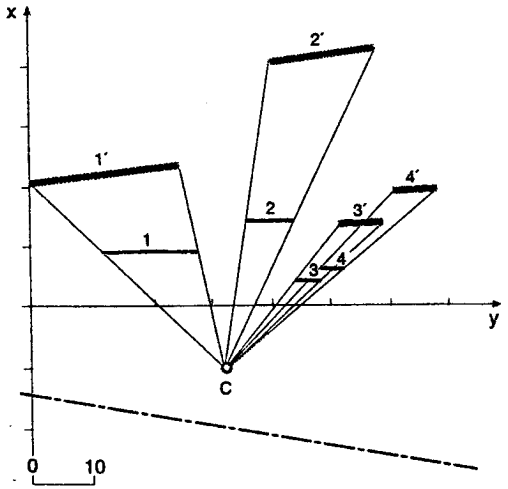


Fig. 3.20a

–0.36	20.75	25.05	23.01
40.34	40.06	58.18	42.37
51.72	13.01	58.76	13.42
60.18	18.18	67.89	18.71

ORD 1 2 4
3

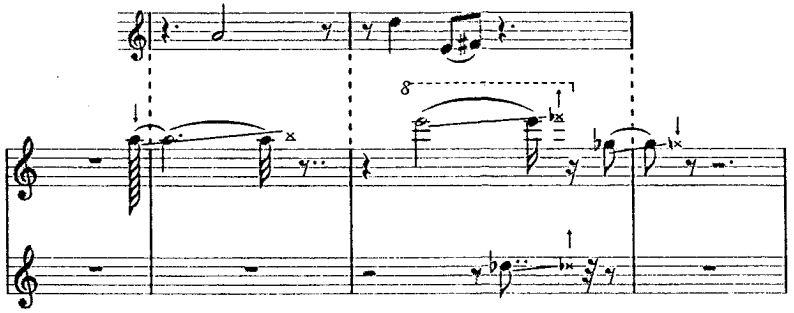


Fig. 3.20b

$q = 0.3$

$0.607463 (\alpha / \alpha + 1) > q > 0$: F is reversed and inverted; sound order is inverted: 4 3 2 1. F "comes" from "south" infinite and tends to settle down on the axis as $q \rightarrow 0$; sounds 4' and 3' partially overlap (Fig. 3.21a, b)

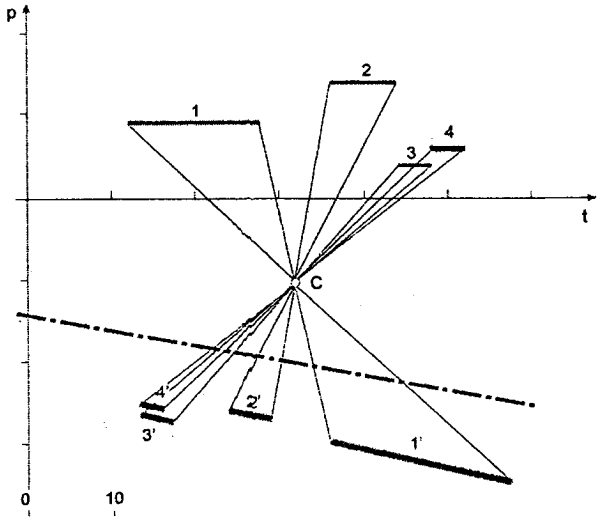


Fig. 3.21a

57.53	–34.25	36.07	–29.87
29.2	–26.79	24.14	–25.7
17.5	–26.9	13.76	–25.95
16.27	–25.72	13.25	–24.99

ORD 4 2 1
3

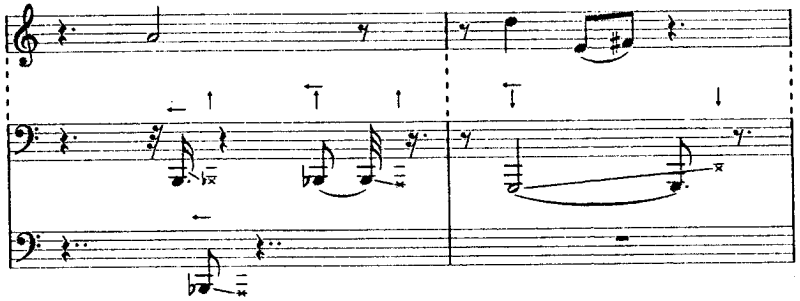


Fig. 3.21b

$q = 0.1$

0.607463 ($\alpha / \alpha + 1$), $q \rightarrow 0$: *F* gets further near the axis (Fig. 3.22a, b)

47.47	–24.69	34.58	–22.28
30.11	–21.3	26.66	–20.66
23.11	–20.36	20.68	–19.9
21.96	–20.03	19.93	–19.65

ORD 4 3 2 1

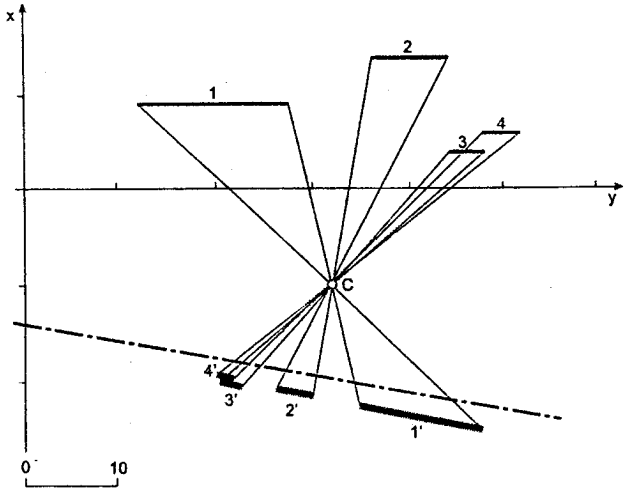


Fig. 3.22a

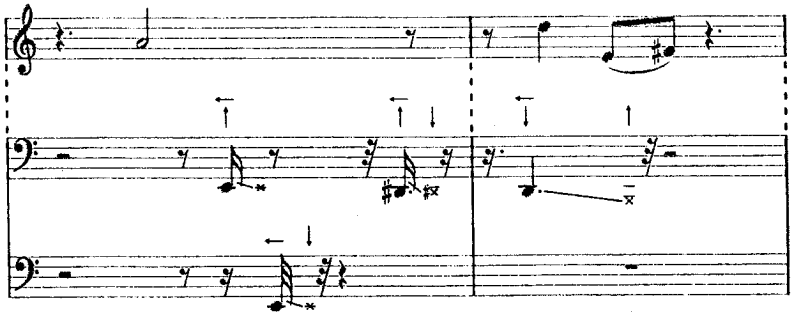


Fig. 3.22b

$q = 0$

F sounds settle down on the axis (Fig. 3.23a, b)

44.92	–22.27	34.18	–20.38
30.38	–19.71	27.4	–19.19
24.55	–18.68	22.48	–18.32
23.49	–18.5	21.75	–18.19

ORD 4 2 1
3

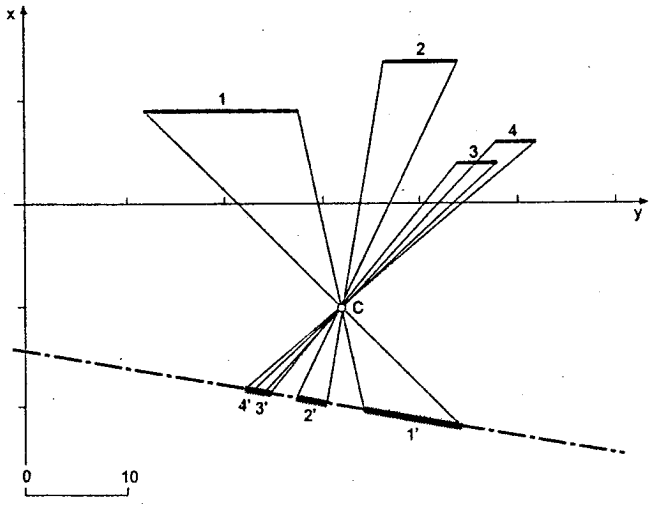


Fig. 3.23a

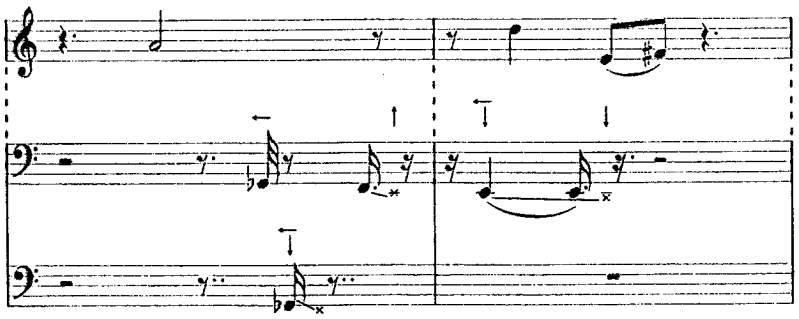


Fig. 3.23b

$q = -0.2$

$-\infty < q < 0$: F tends to the center as $q \rightarrow -\infty$ (Fig. 3.24a, b)

41.72	–19.23	33.66	–17.93
30.73	–17.58	28.4	–17.19
26.37	–16.56	24.78	–16.31
25.49	–16.5	24.13	–16.29

ORD 4 2 1

3

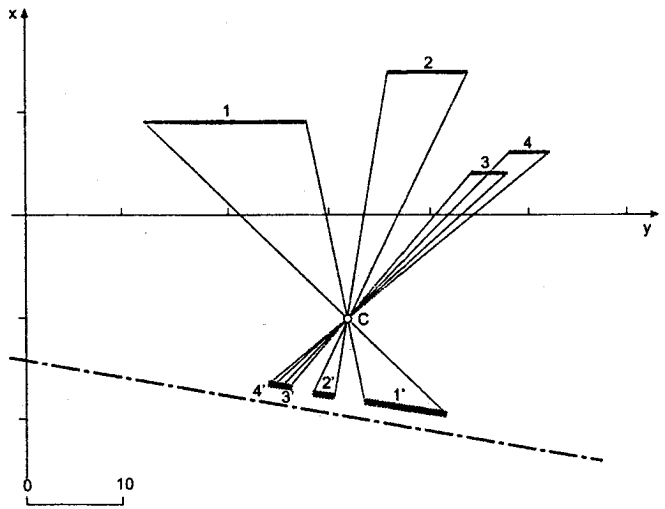


Fig. 3.24a

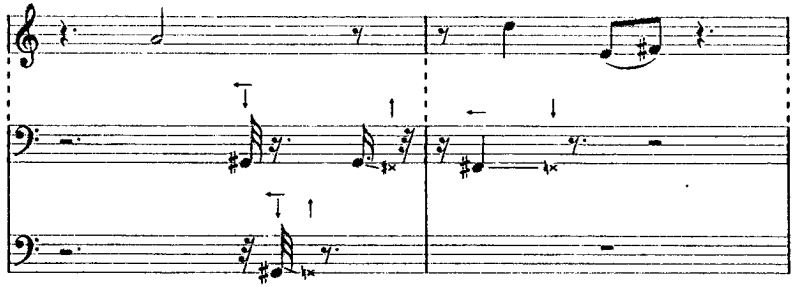



Fig. 3.24b

$q = -2$

$-\infty < q < 0$: F agglomerates further towards the center (Fig. 3.25a, b)

35.01	–12.86	32.53	–12.53
31.57	–12.55	30.77	–12.44
30.24	–12.04	29.72	–11.98
29.90	–12.09	29.45	–12.08

ORD 4 2 1

3

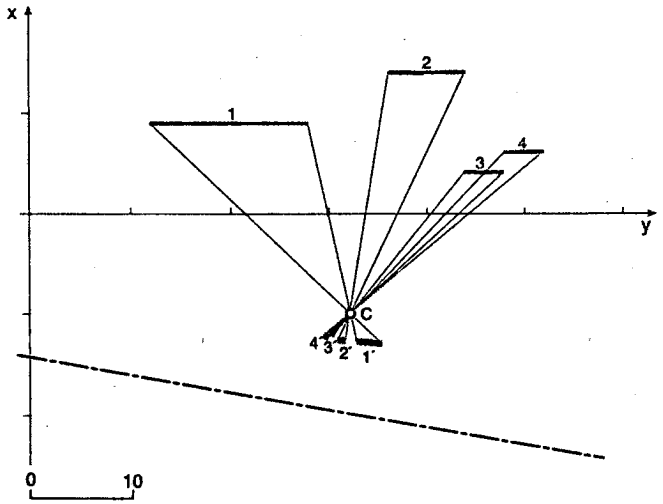


Fig. 3.25a

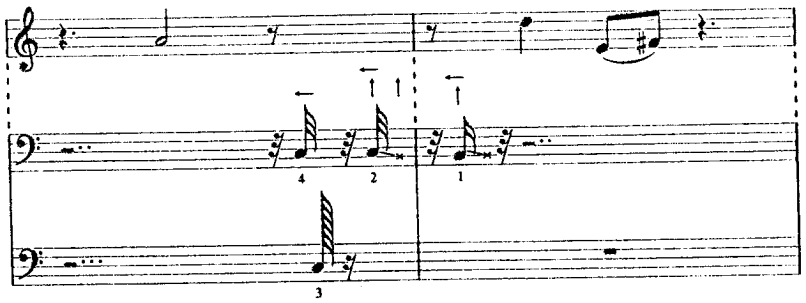


Fig. 3.25b

$q = 1.7$

$+\infty > q > 1$: F' is in "direct" condition (non-reversed and non inverted), tends to its initial form, mingling with it when $q = 1$ (Fig. 3.26a, b)

24.81	–3.17	30.65	–3.62
38.16	–3.	35.4	–3.18
35.24	–5.04	37.56	–5.18
37.3	–4.69	38.52	–4.78

ORD 1 2 3 4

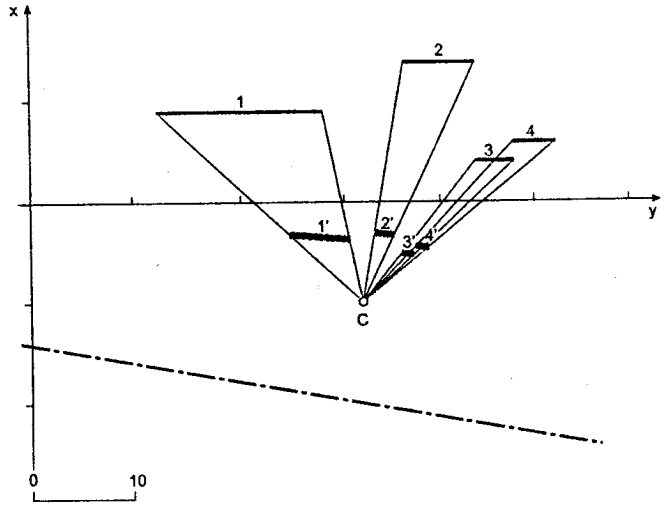


Fig. 3.26a

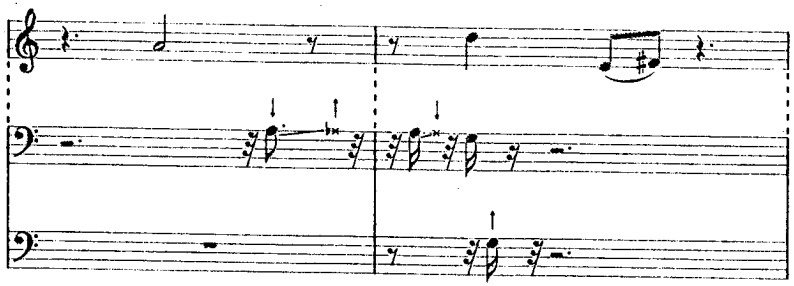


Fig. 3.26b

3.2.5. Parallel homology (center situated at infinity)

We suggest the reader re-examine A.3.6 and A.3.8.

Further on, we will study the behaviour of the F sound form under the action of parallel homology transformation. We remind that parallel homology is a homology whose center is situated at infinity; in this case the projections lines are parallel.

F', the image of F by this transformation depends on the values of the parameters defining the homology axis, the angle β made by the projection lines with the homology axis and the homology ratio q.[1] Unlike the homology with center at finite distance, q may take continuous values on the $(-\infty, +\infty)$ interval.

For $q = 1$, we obtain the identical transformation. For $1 < q < +\infty$, F expands. For $0 < q < 1$ it compresses. For $q = 0$, F sets down on axis. For $0 > q > -1$, F appears compressed and reversed. For $q = -1$, F' is symmetrical with F related to the axis; for $-\infty < q < -1$, F' is reversed and dilated as regards F.

As working method, the behaviour of F is studied as related to an axis defined by the point P of coordinates 32, 0 and the angle $\alpha = -10^\circ$ made by the homology axis with the Ox axis, while the angle β of the projection lines with the homology axis takes successively the 45°, 90° and 135° values.

Some examples regarding the behaviour of F are further given when the homology axis is defined by (a) P of coordinates (0, 2), $\alpha = 0^\circ$; (b) P of coordinates (12, 0), $\alpha = 45^\circ$, successively. With case (a), the angle β is considered as successively having 45°, 90° and 135°; and with case (b), β has 90°.

---

[1] It is worth mentioning again that the center C being situated at infinity, $CM = CM'$, and, consequently, the homology biratio

$$\frac{CM}{MM_1} / \frac{CM'}{M'M_1}$$ becomes the $$\frac{MM_1}{M'M_1}$$ ratio (A.3.6).

3.2.5a: *Classical methods of composition revealed as special cases of parallel and orthogonal homologies*

Various values of the ratio q are used for each case, so that the behaviour of F is surveyed in the different considered cases. It is emphsized by the fact that the reversion, inversion, increase and decrease of some musical fragment are particular cases of transformations through parallel homology. In this connection, at the end of this chapter, the transformation obtained by $q = -1$ is investigated. For this particular value of q when the projections lines are orthogonal on the axis ($\beta = 90^\circ$) and the axis is parallel to the time axis ($\alpha = 0^\circ$), the parallel homology produces the classical reversion of a musical fragment (Fig. 3.37), and when the axis is perpendicular to the time axis ($\alpha = 90^\circ$) it causes the classical inversion of such a fragment (Fig. 3.38).

Finally, in this last case, for $q = 2$ we obtain the increase of a musical fragment (doubling of duration value) (Fig. 3.39), for $q = 0.5$ we obtain the decrease of the fragment (bisection of duration value) (Fig. 3.41), both being usual methods in classical polyphony.

For $q = 1.5$ we obtain the transformation introduced by Olivier Messiaen: point ajouté (added point), i.e. the increase of a fragment by adding half of its initial value (Fig. 3.40).

Any "added value" (in the Messiaen sense – 1.5.8) can be expressed by a ratio of such a homology; e.g. *ûû + çç → q = 1.126; ïïï + çç → q = 1.0615; ïïï + öööö + çç → q = 1.3125.*

$\alpha = -10$; $\beta = 45^\circ$; $q = 2.5$

$1 < q < +\infty$: The sounds of F extend and tend to infinity in the direction of the projection lines as $q \to +\infty$. From a musical point of view, the duration and pitch interval values increase (Fig. 3.27a, b)

21.36	15.55	42.19	18.93
61.46	31.62	71.57	33.31
54.46	11.32	59.67	12.17
63.09	16.57	68.3	17.41

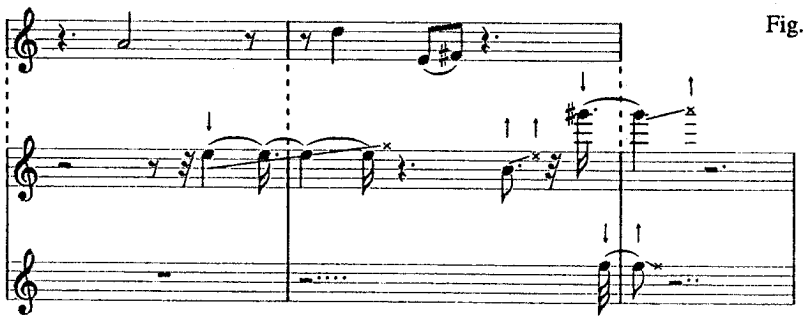



Fig. 3. 27b

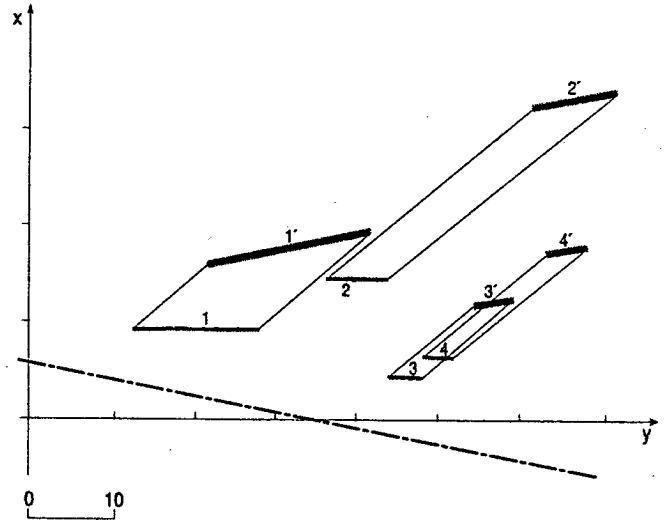


Fig. 3.27a

$q = 0.75$

$0 < q < 1$: F compresses tending towards the axis, as $q \to 0$ (Fig. 3.28a, b)

10.43	7.9	25.63	7.34
31.8	11.06	39.4	10.78
42.25	2.77	46.05	2.63
45.48	4.23	49.28	4.09

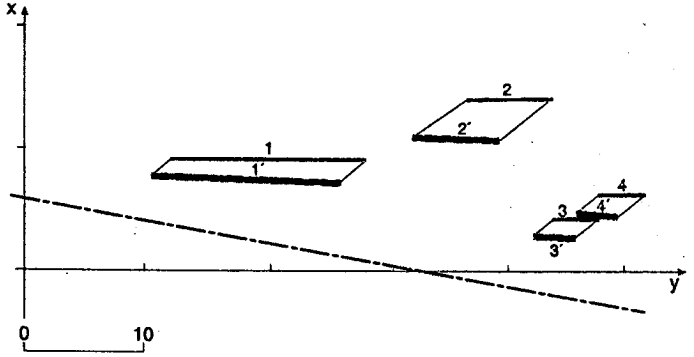


Fig. 3.28a

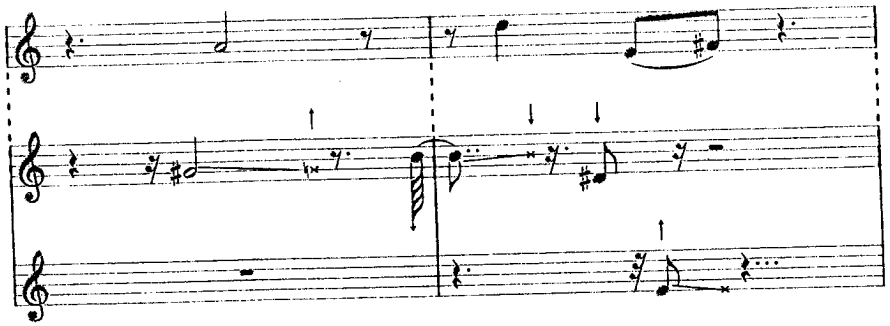



Fig. 3.28b

$q = -0.5$

$-1 < q < 0$: F' is reversed, and tends to become symmetrical with F as $q \to -1$ (see Fig 3.35, 36, 37) (Fig 3.29a, b)

2.6	2.44	13.8	–0.9
10.83	–3.62	16.42	–5.31
33.53	–3.32	36.32	–4.17
32.9	–4.57	35.69	–5.41

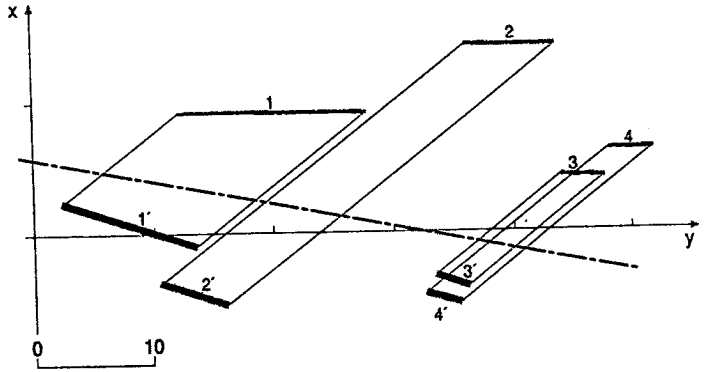


Fig. 3. 29a

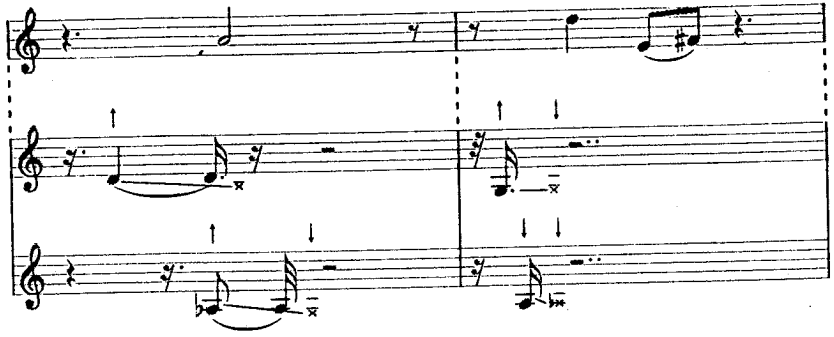



Fig. 3. 29b

$q = -3$

$-\infty < q < -1$: F' expands to infinity in the projection sense as $q \to -\infty$ (Fig. 3.30a, b)

–12.97	–8.48	–9.8	–17.5
–31.1	–32.98	–29.54	–37.49
16.09	–15.54	16.87	–17.79
7.74	–22.18	8.52	–24.44

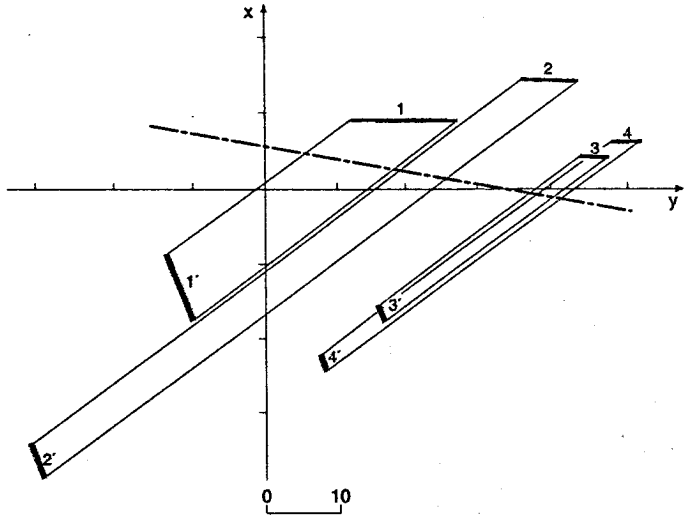


Fig. 3.30a

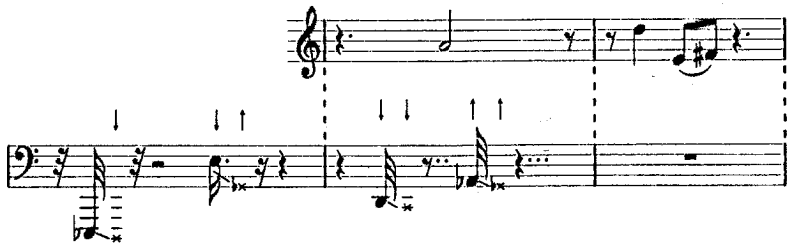



Fig. 3.30b

$q = -0.75$

Projection slope $\beta = -135^\circ$ (Fig. 3.31a, b)

19.65	–1.92	39.59	–7.56
56.55	–15.35	66.52	–18.17
52.54	–8.21	57.53	–9.61
60.33	–11.61	65.31	–13.01

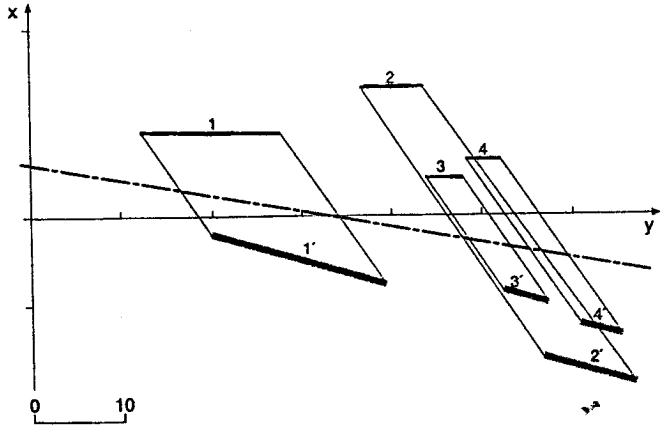


Fig. 3.31a

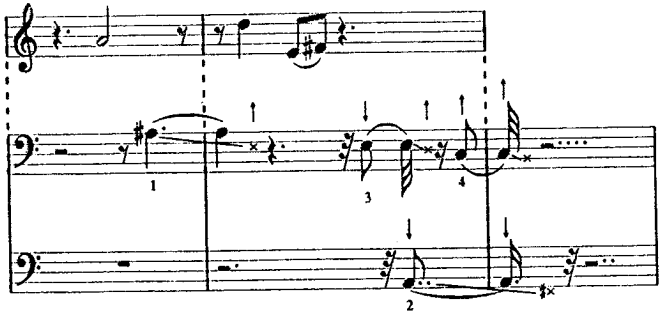



Fig. 3.31b

$q = 2$

The same projection slope as in Fig. 3.30 ($\beta = 45°$), but $\alpha = 0°$: parallel axis to time axis. Since sounds are parallel to the homology axis, no longer "glissando" sounds; their duration is maintained. Pitch interval dilation (Fig. 3.32a, b).

19	16	35	16
48	26	56	26
46	6	50	6
52	10	56	10

Fig. 3. 32a

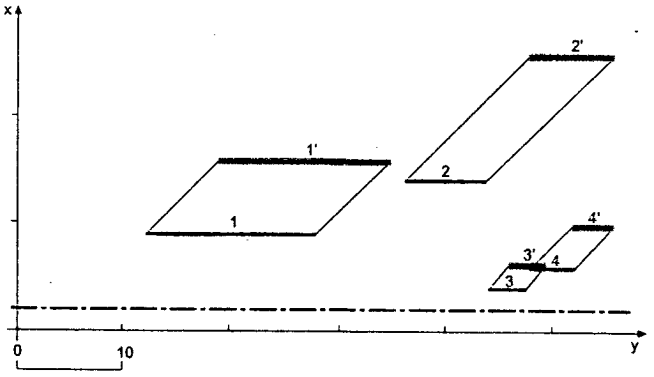
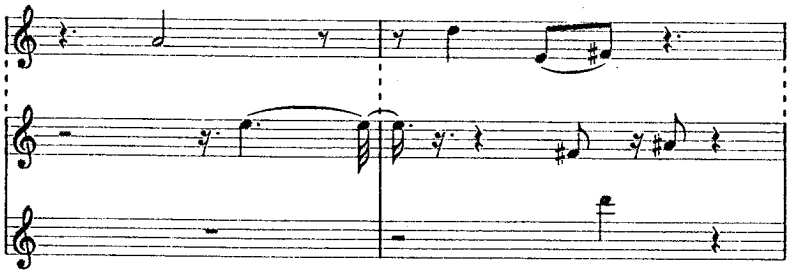


Fig. 3.32b

$q = -2$

Projection lines slope 90°: *Augmented reversion*! (Fig. 3.33a, b)

12	–12	28	–12
36	–22	44	–22
44	–2	48	–2
48	–6	52	–6

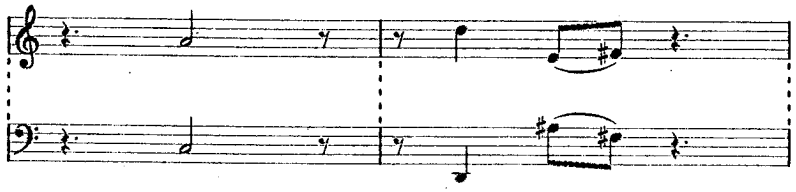
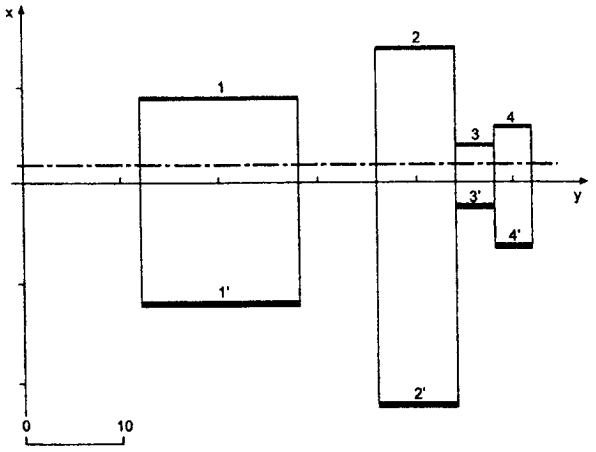



Fig. 3. 33b

Fig. 3.33a



$q = 2$

Projection lines slope135°: F' sounds anticipate the corresponding F sounds. Pitch interval dilation (Fig. 3.34a, b)

5	16	21	16
24	26	32	26
42	6	46	6
44	10	48	10

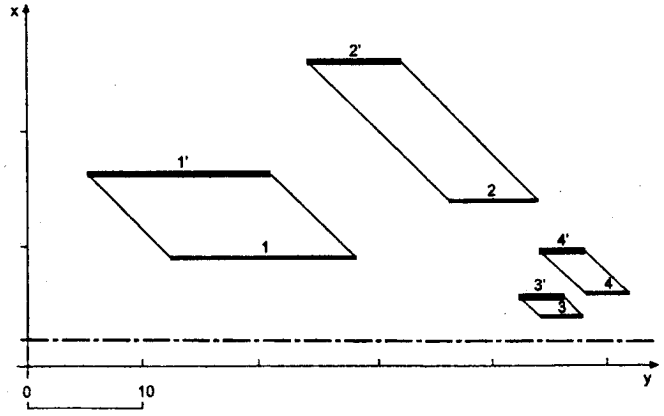


Fig. 3.34a

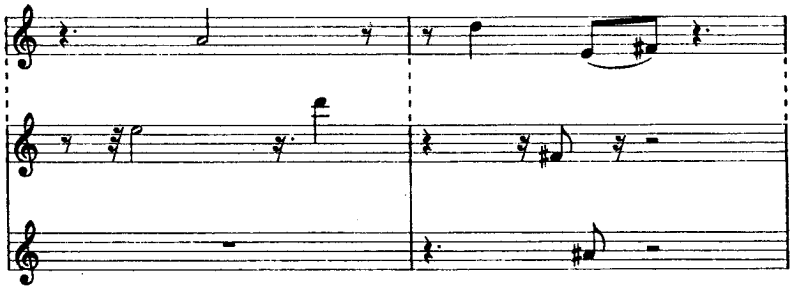


Fig. 3.34b

3.2.6. Transformations through parallel homology of $q = -1$ value, which in certain situations of the axis and projections (A.3.8), cause the reversion and inversion used in music (2.4.4, 2.4.5)

Figs. 3.35a, b illustrate the transformation with ratio $q = -1$ when $\alpha = -10°$, $\beta = 45°$; when $\alpha = -10°$, $\beta = 90°$, Fig. 3.36a, b; when $\alpha = 0°$, $\beta = 90°$, Fig. 3.37a, b; we obtain the classical symmetrical reversion related to the musical sound D3, axis ordinate $y = 2$.

Fig. 3.38: $\alpha = 90°$, $\beta = 90°$ the transformation causes the inversion of the fragment F.

$q = -1$: F' is symmetrical with F related to the axis (Fig. 3.35a, b)

–0.48	0.25	9.07	–4.25
2.44	–9.49	7.22	–11.74
30.04	–5.77	32.43	–6.89
27.87	–8.09	30.26	–9.22

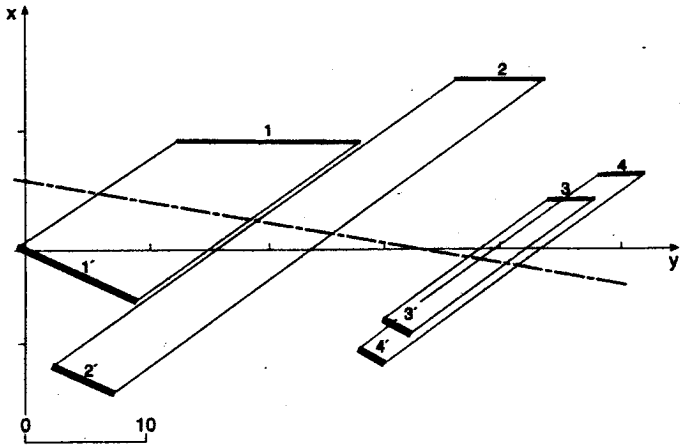


Fig. 3.35a

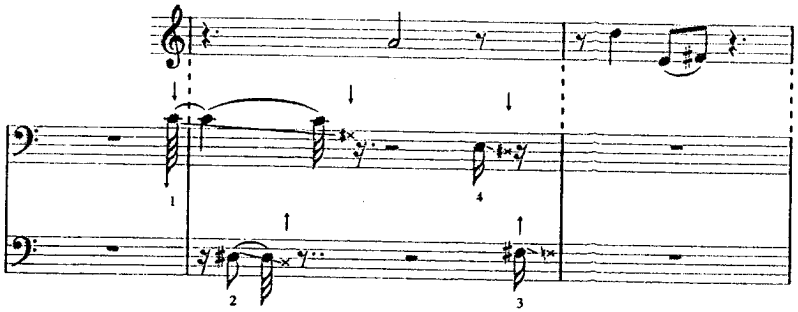



Fig. 3.35b

10.12	–1.61	25.16	–7.08
30.97	–14.52	38.48	–17.25
41.9	–7.86	45.66	–9.23
44.93	–11.11	48.74	–12.47

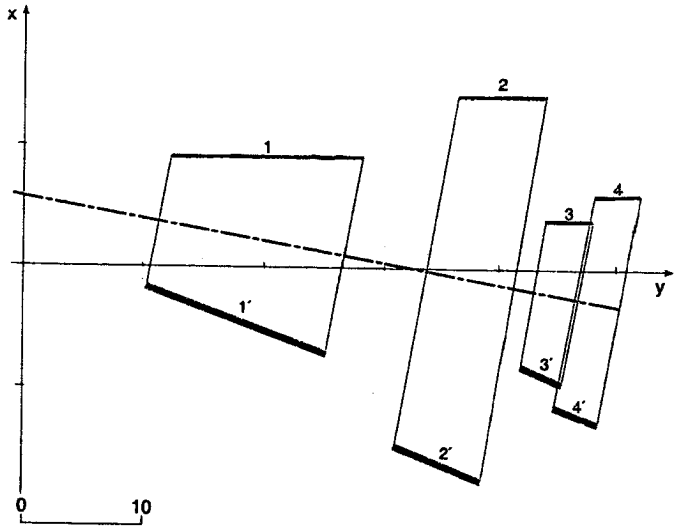


Fig. 3.36a

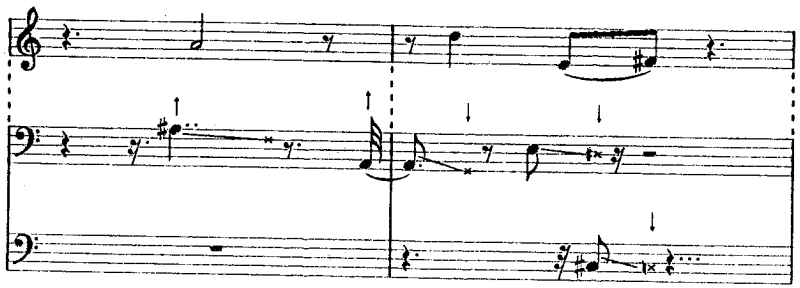



Fig. 3.36b

$\alpha = 0°$; $\beta = 90°$; $q = -1$: Classical reversion axis, the musical sound D3 (1.5.7d; 2.4.3d)

12	–5	28	–5
36	–10	44	–10
44	0	48	0
48	–2	52	–2

Fig. 3.37a

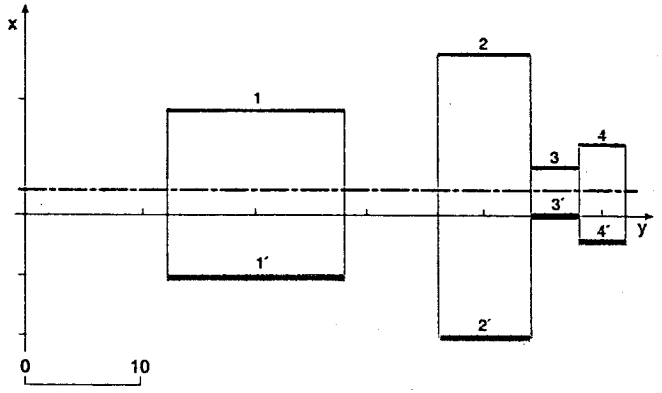
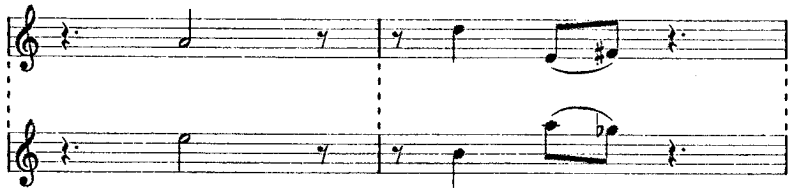


Fig. 3.37b

Perpendicular axis to the time axis, orthogonal projections ($\alpha = 90°$; $\beta = 90°$) $q = -1$; inversion occurs

12	9	–4	9
–12	14	–20	14
–20	4	–24	4
–24	6	–28	8

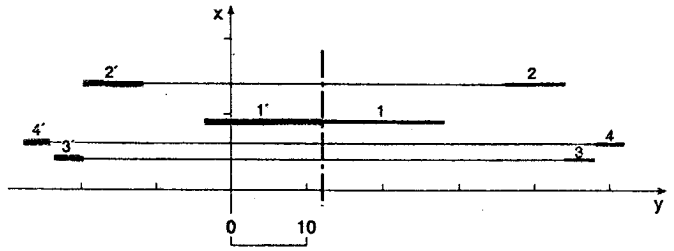


Fig. 3.38a

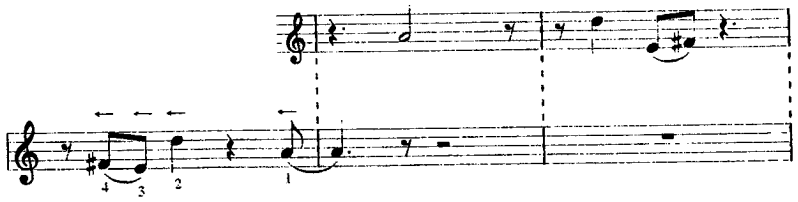



Fig. 3.38b

Classical inversion! (1.5.7d; 2.4.3d)

Figs. 3.39a and b and Figs. 3.40a and b illustrate, respectively, the fragment increase (duration doubling, $q = 2$), increasing by half the initial value (point ajouté) $q = 1.5$ (1.5.8); Fig. 3.39 illustrates the decrease through values bisection, $q = 0.5$. We now let the reader imagine increases and decreases of any values of q ratio, from $q = 0$, when all the sounds agglomerate on the homology axes (duration 0, null sounds). To $q \rightarrow +\infty$, when the fragment tends to have an infinite length.

$\alpha = 90°$, $\beta = 90°$; $q = 2$

12	9	44	9
60	14	76	14
76	4	84	4
84	6	92	6

Classical doubling of duration values (1.5.8; 2.4.3a)

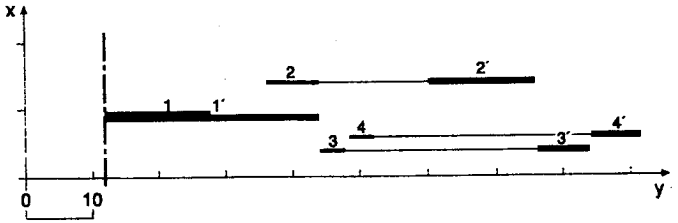


Fig. 3.39a

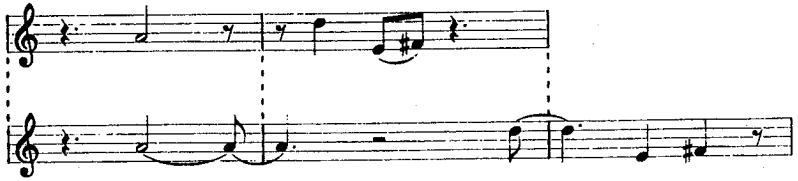



Fig. 3.39b

$\alpha = 90°$, $\beta = 90°$; $q = 1.5$.

12	9	36	9
48	14	60	14
60	4	66	4
66	6	72	6

Increase by half of the value of durations, method referred to as "point ajouté" (increase by adding a point to each value (Olivier Messiaen)) (1.5.7).

Fig. 3.40a

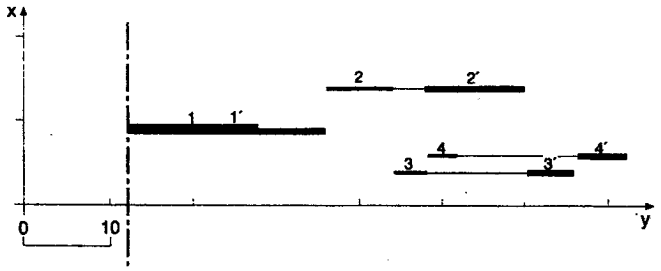
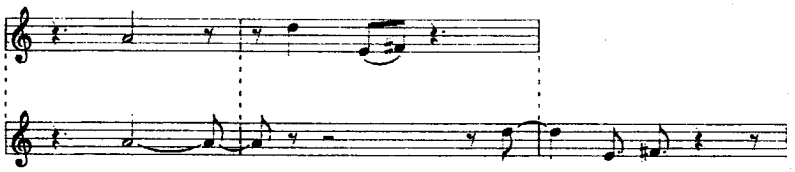


Fig. 3.40b

$\alpha = 90°$ (axis perpendicular to the time axis), $\beta = 90°$ (orthogonal projection), $q = 0.5$:

12	9	20	9
24	14	28	14
28	4	30	4
30	6	32	6

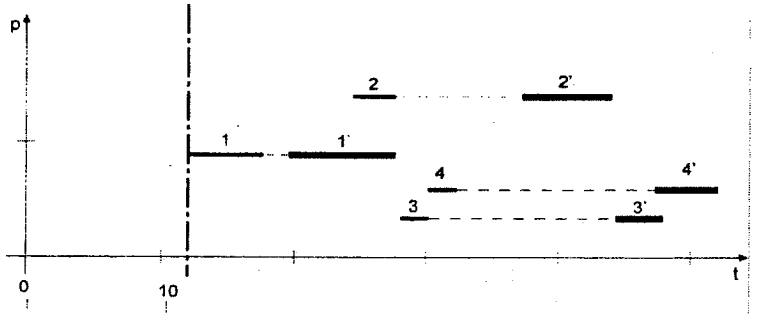


Fig. 3.41a

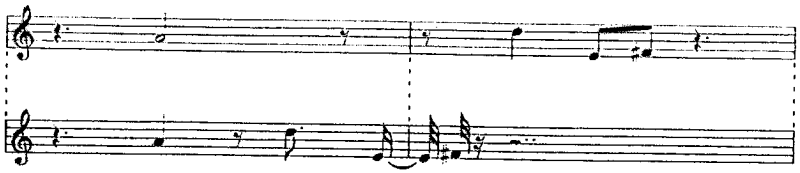



Fig. 3.41b

Duration classical bisection! (1.5.8; 3.2.5)
Classical decrease by value bisection.

3.2.7. A story with J. S. Bach, fugues "in mirror" and homologies

"Papa" Bach discovers that in his works he uses homologies defined by certain parameter values[2] and begins to show an interest in giving other values to these parameters.

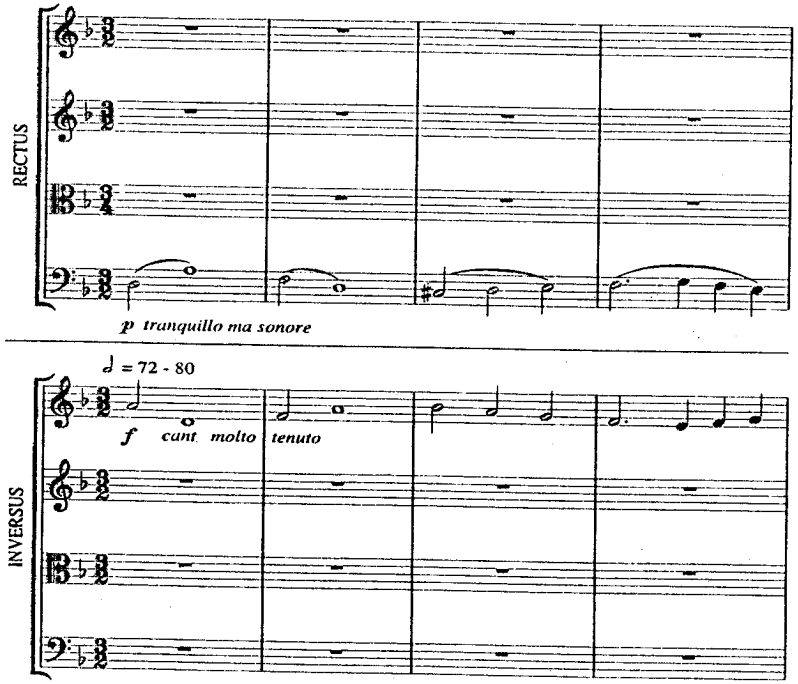



Fig. 3.42a

[2] In Contrapunctus XII from "The Art of Fugue" (out of which we present the first 12 measures in Fig. 3.42) it is obvious that the "Inversus" fugue is a key symmetry of the "Rectus" fugue. According to (2.3.9), the key symmetry may be interpreted as a particular case of homology composed with modal adjustement (Fig. 3.42).

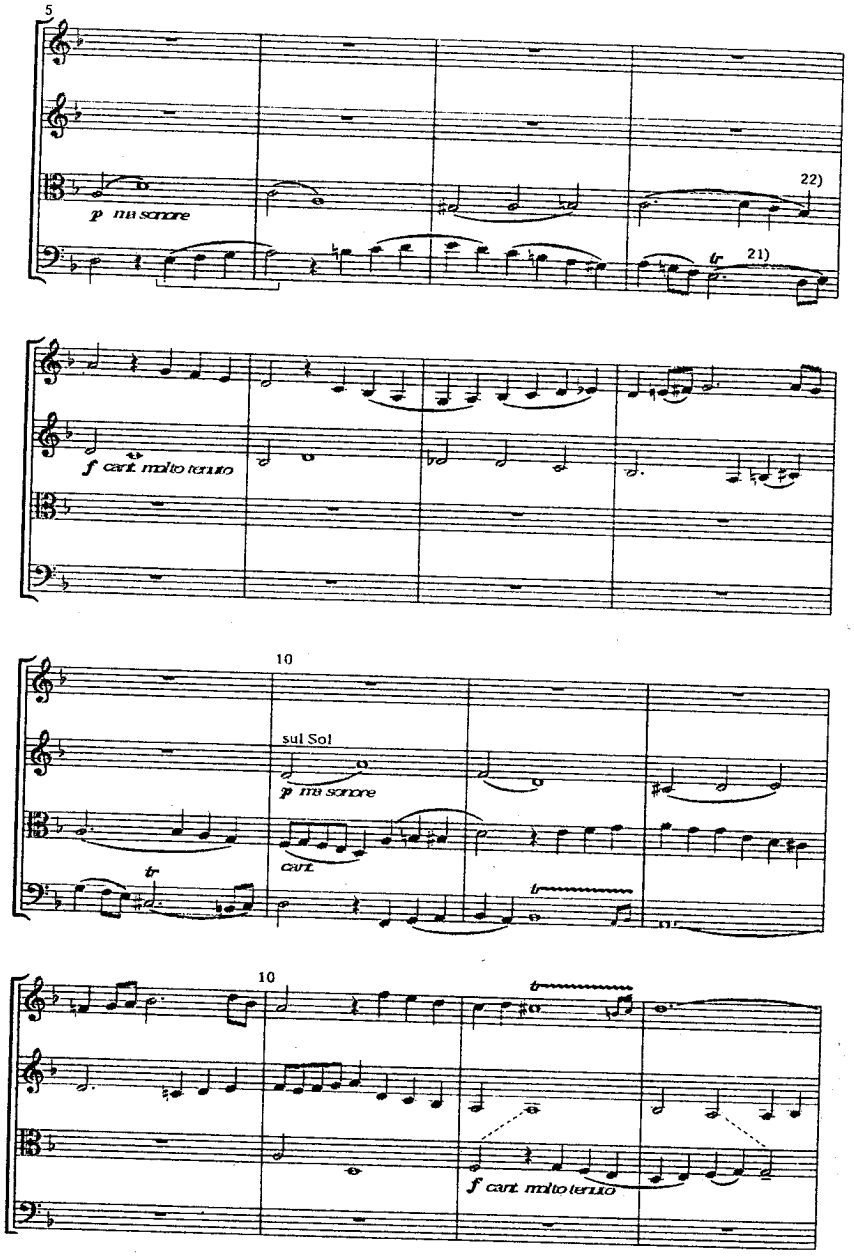



Fig. 3.42b

For reasons of space, we will confine outselves to the first 4 measures of the fugue. Of course, the methods we have in view when treating them may be applied to the entire text of the fugue (Fig. 3.42a, b).

In order to convert through homology the "Rectus" to the "Inversus" which is situated at the upper quintoctave interval (D2 (y = –10) – A3 (y = 9) = 19 semitones), the homology axis (parallel to the time axis) should divide this interval into two equal intervals of 9 1/2 semitones each.

Consequently, the homology axis will pass through the point of coordinates $x = n$, $y = -0.5$.

According to (A.3.13; 2.4.3d; 3.2.5), $\alpha = 0°$, $\beta = 90°$, $q = -1$ (Fig. 3.43).

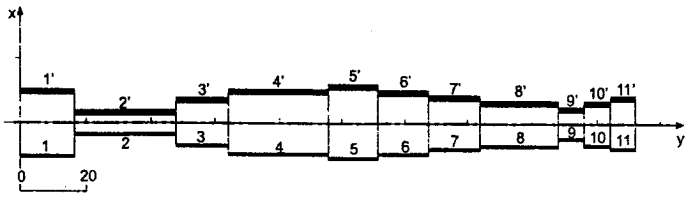


Fig. 3.43

By compounding this transformation with a modal adjustement (**MA**.2.3.9; 3.1.2) by translating F#3 (y = 6) → F# (y = 5), we can obtain the transformation made by J. S. Bach (Fig. 3.44).

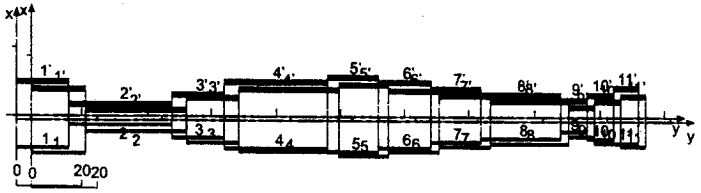


Fig. 3.44

"Papa" Bach was peering with wisdom over my shoulder at the demonstration, advising me to show him what will happen when, with the same axis and the same projections, q successively takes the 2, 0.5 and –1.2 values. I show him the results from the computer: Figs. 3.45, 3.46 and table on Fig. 3.47, exhibiting, in musical notation, the transformations obtained when q takes the values demanded by him, to which I have added the following ones 1.6, 1.3, –0.5, –1 + modal adjustement (2.3.9; 3.1.2) (Fig. 3.45).

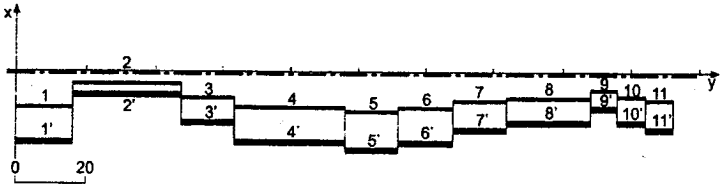


Fig. 3.45

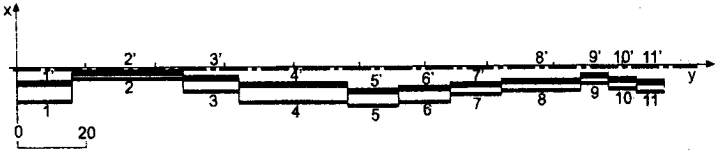


Fig. 3.46

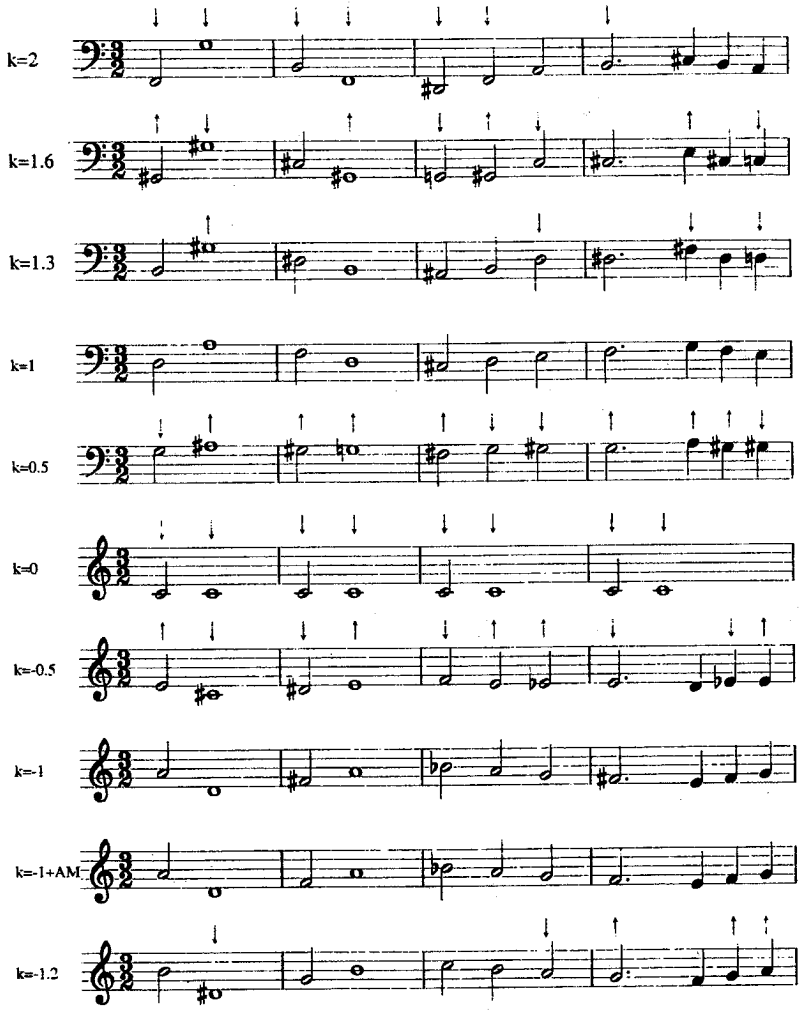



Fig. 3.47

My venerable friend becomes more and more interested: now he would like to see his theme transformed with the same values of q, but he wants the

angle α of the homology axis with the time axis to be 3° and the angle of the projections on the axis to be $\beta = -45°$.

The computer shows the homeomorphisms for q = –1.0, 0.5 and 2 (Figs. 3.48, 49, 50, 51). Then it displays in musical notation all the required transformations (Fig. 3.52a, b).

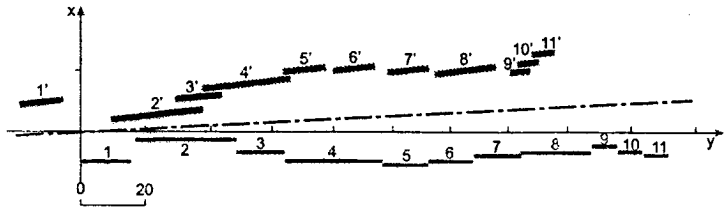


Fig. 3.48

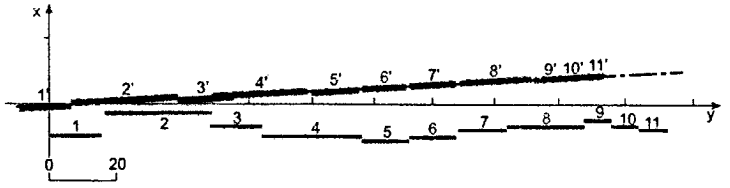


Fig. 3.49

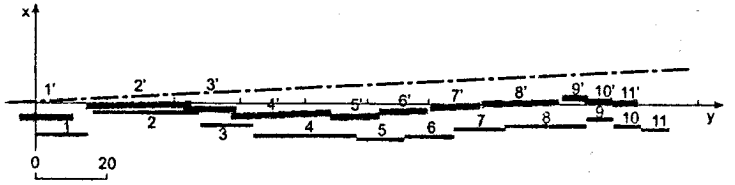


Fig. 3.50

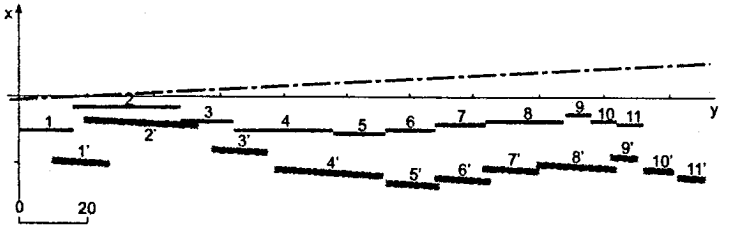


Fig. 3.51

The shadow behind me is more and more fascinated. Now, he wants to bring the homology center to a finite distance and selects the x = 96 and y = –250 coordinates.

The transformations for q = 2, 0.5, 0, –0.5, –1 and –1.2 are displayed on the screen (Figs. 3.53, 54, 55, 56, 57, 58).

The table in Fig. 3.52a, b (representing the transformation when the center is at –∞) becomes the table in Fig. 3.59.

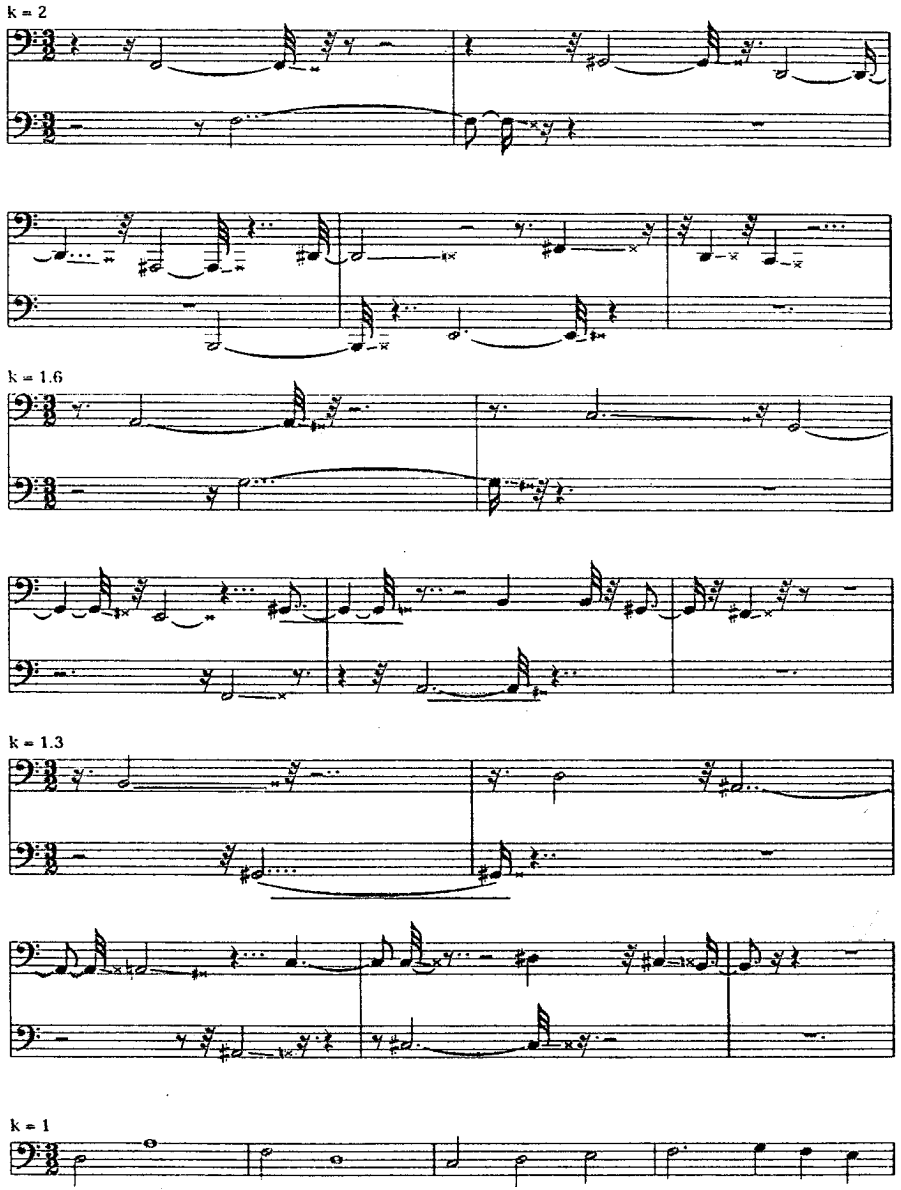


Fig. 3.52a

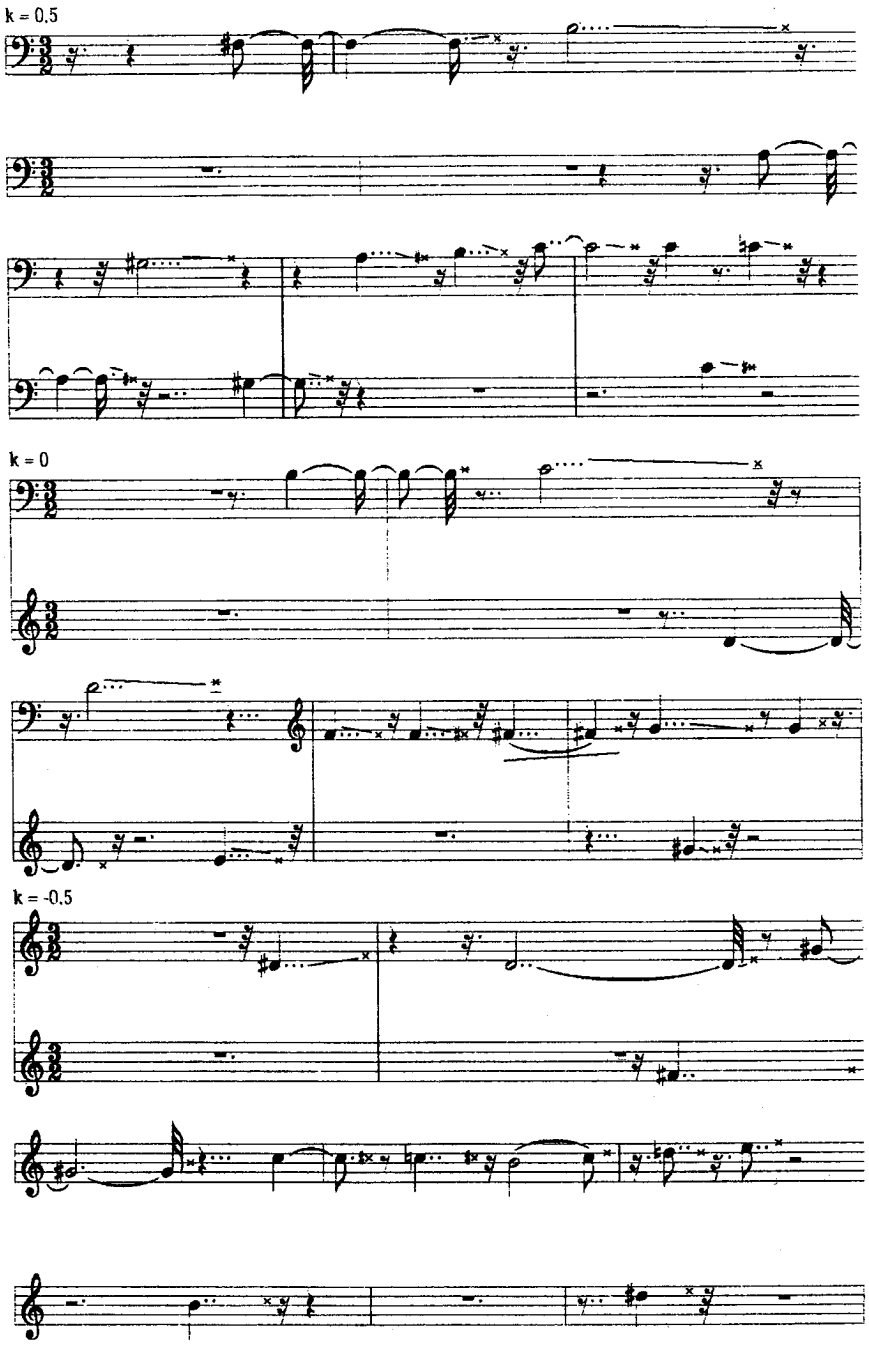



Fig. 3.52b

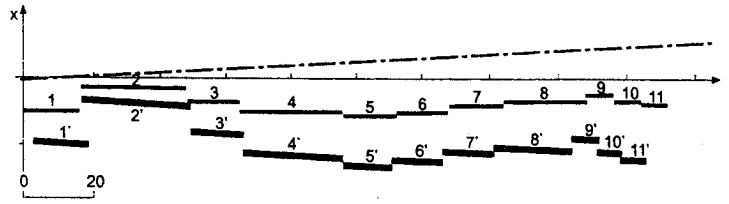


Fig. 3.53

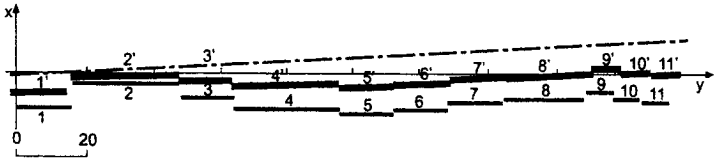


Fig. 3.54

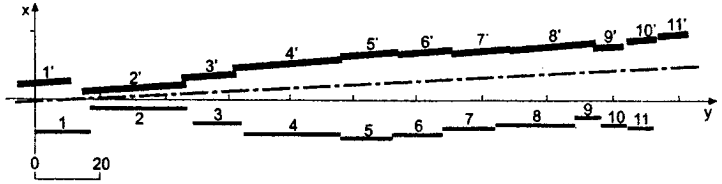


Fig. 3.55

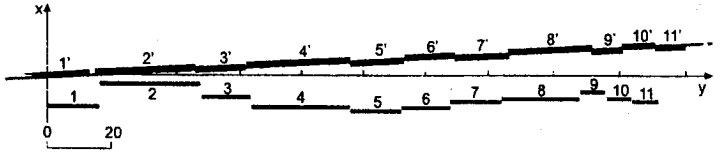


Fig. 3.56

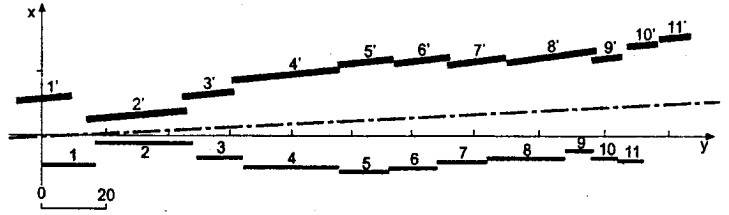


Fig. 3.57

The veteran of polyphony, realizing that the preceding example was only case I of the homology with its center at finite distance (object situated between center and axis), has one more question: what would happen to the theme

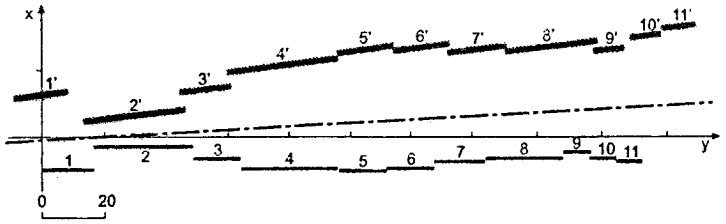


Fig. 3.58

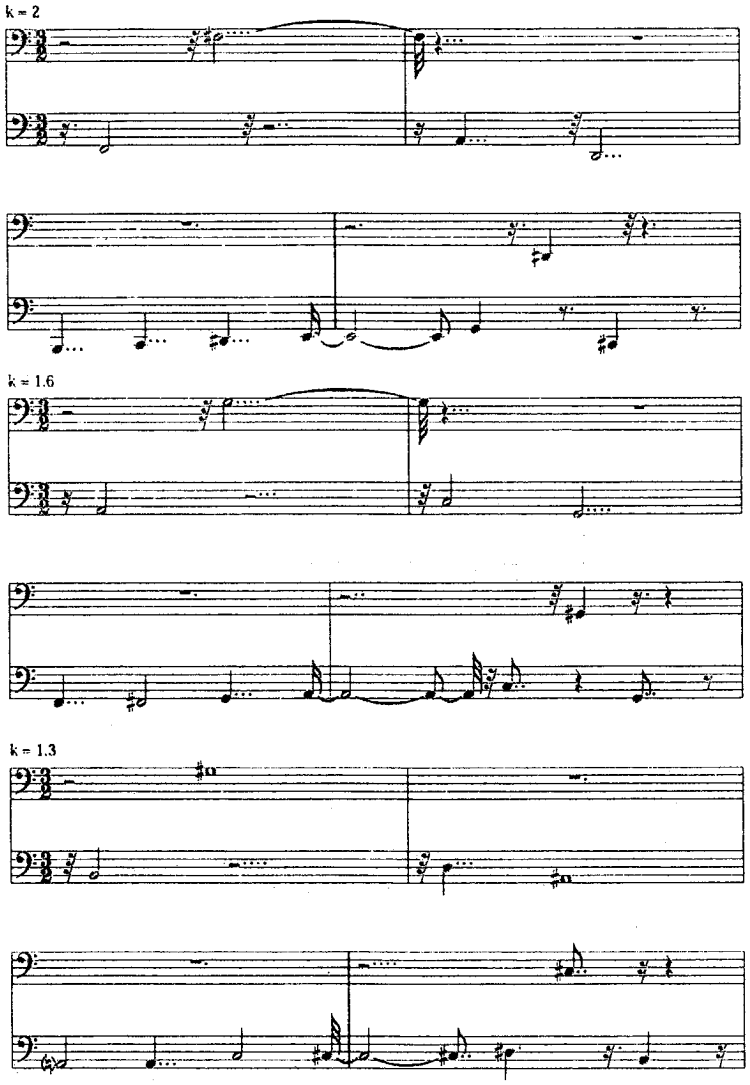


Fig. 3.59

of his fugue when treated in the key of the two other cases (axis between object and center and center between object and axis)? Out of the infinity of modes the axis and the center can be arranged and given values to the biratio q, he selects for case II the axis defined by the point $x = 200$, $y = -2$ and angle $\alpha = 0°$, the center situated in point $x = 140$, $y = 19$ and the biratio $q = -50, -10$, and, for the case III the axis $x = 0$, $y = 3$; the center $x = 50$, $y = 0$, $q = 0.87, -0.5, -1$.

The transformations obtained by these steps are submitted to him both in the sound space $\mathbf{R}^2$ and in musical notation (Figs. 3.60, 61, 62, 63, 64, 65).

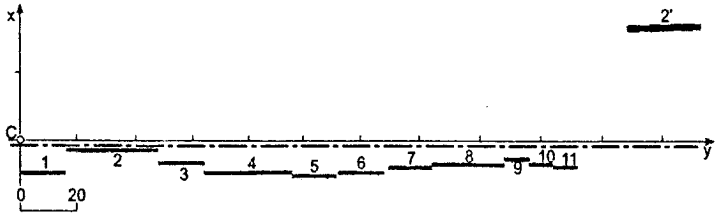


Fig. 3.60

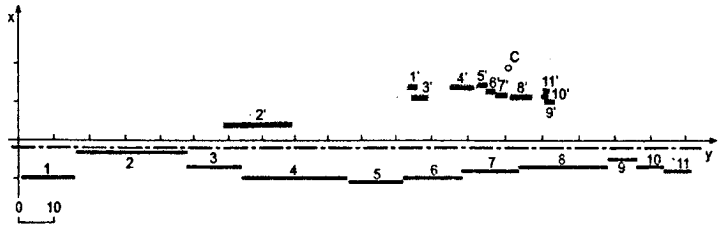


Fig. 3.61

The Great Master contemplates the last results and for a long time muses upon them. Then he walks away to his working desk, sits down and goes on dictating the "Inversus" fugue from the Contrapunctus XII.

The fact that the method used by him is a particular case of homology does not seem to impress him much; he leaves this concern to the musicians to come three centuries later.

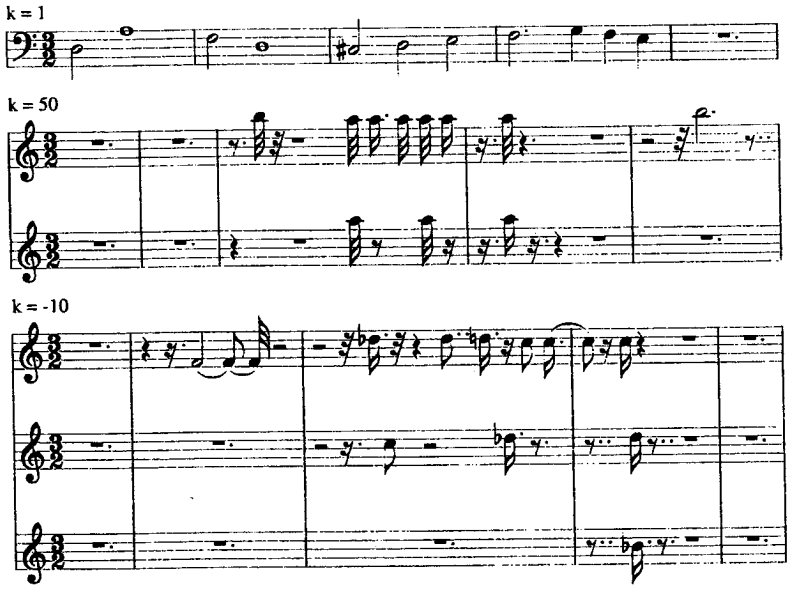



Fig. 3.62

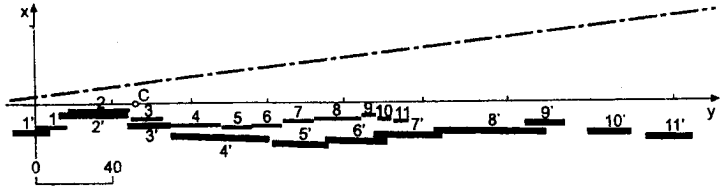


Fig. 3.63

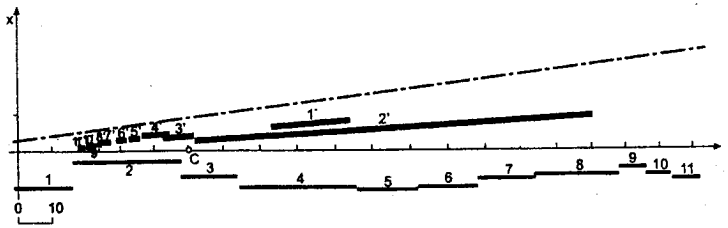


Fig 3.64

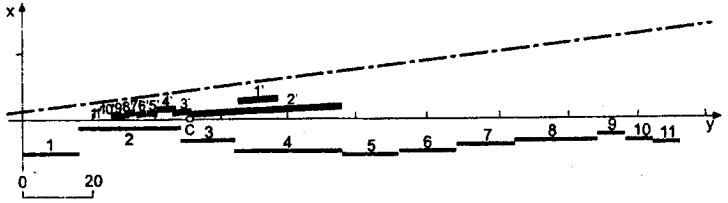


Fig. 3.65

3.3. BEHAVIOUR OF SOUND FORMS UNDER THE ACTION OF TRANSFORMATIONS THROUGH HOMOTHETY

We suggest the reader re-examine A.2 and A.3.15.

3.3.1. A part of the main theme of George Enescu's First Symphony subjected to homothety

Let F be a sound form consisting of several musical sounds, and Co a homothety center. According to (A.3.15), the images of F through homothety depend on the value of ratio k. For $1 < k < +\infty$, F' expands, moves in the opposite direction with respect to F and Co. For $k = 1$ we obtain the identical image of F. For $1 > k > 0$, F is compressed: the sounds heap up towards the homothety center as $k \to 0$. For $k = 0$, the entire form is compressed in the center C (the topological character of this transformation is compromised).

For $0 > k > -1$, F is inverted, reversed and compressed; for $k = -1$, F is inverted and reversed; $k = -1$ causes central symmetry (A.3.15; 1.5.7). From a musical point of view, we obtain in this case the "inversion's reversion". Finally, for $-1 > k$, F is inverted and reversed, the sounds increase in duration and migrate toward ∞.

In the following study we will consider as sound form a part of the main theme of George Enescu's First Symphony defined by the matrix

0	3	8	3
8	15	16	15
16	10	24	10
24	17	36	17
36	19	40	19
40	15	44	15
44	17	48	17
48	21	64	21

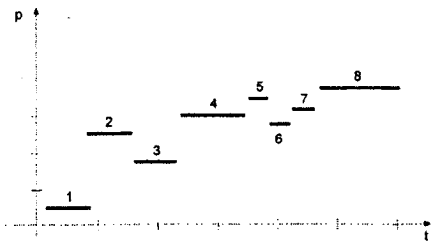
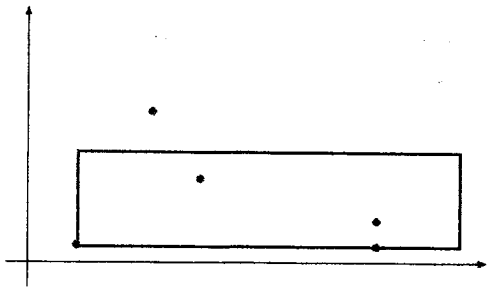


Fig. 3.68a

and represented in sound space $\mathbf{R}^2$ (Fig. 3.68a).

As investigating method, we will consider the homothety center successively situated outside the rectangle defining the sound form space (1.1.5), $(x,y)Co = 35, 16$ and inside it $(x,y)Co = 5, 16$ (Fig. 3.69).

Fig. 3.69



For each position of the center, k will take the values 1.7, 0.5, –0.5, –1, –2.

C (35, 16)

$k = 1.7$

–24.50	–6.10	–10.90	–6.10
–10.90	14.30	2.70	14.30
2.70	5.80	16.30	5.80
16.30	17.70	36.70	17.70
36.70	21.10	43.50	21.10
43.50	14.30	50.30	14.30
50.30	17.70	84.30	22.80

ORD 1 2 3 4 5 6 7 8

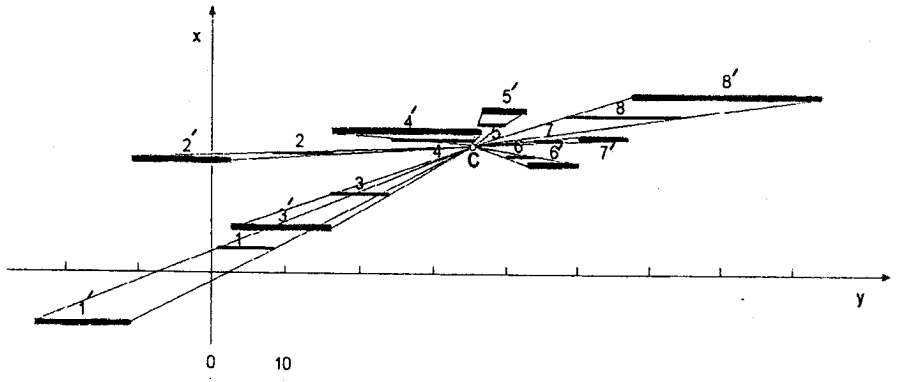


Fig. 3.70a

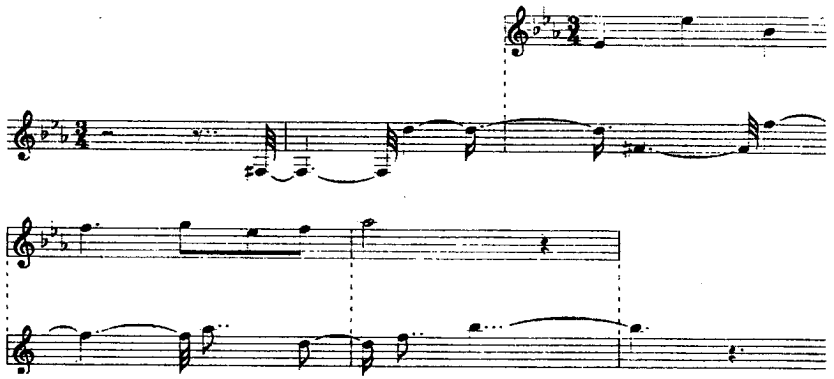



Fig. 3.70b

$k = 0.5$

17.50	9.50	21.50	9.50
21.50	15.50	25.50	15.50
25.50	13.00	29.50	13.00
29.50	16.50	35.50	16.50
35.50	17.50	37.50	17.50
37.50	15.50	39.50	15.50
39.50	16.50	41.50	16.50
41.50	18.00	49.50	18.00

ORD 1 2 3 4 5 6 7 8

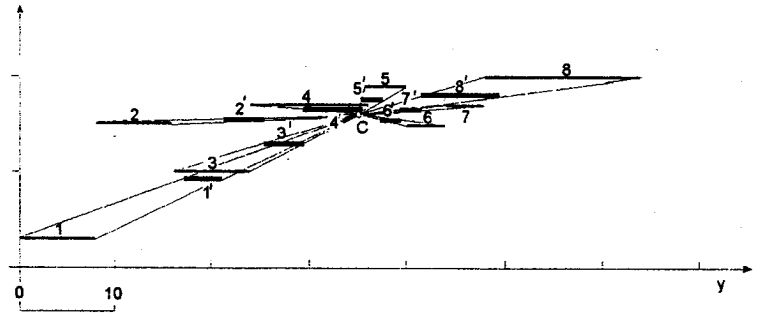


Fig. 3.71a

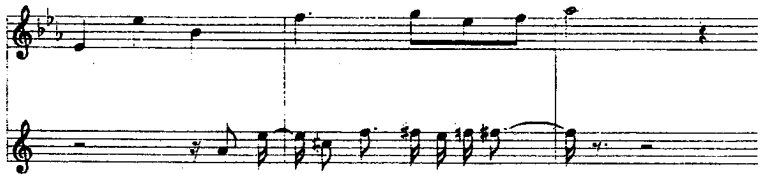



Fig. 3.71b

$k = -0.5$

52.50	22.50	48.50	22.50
48.50	16.50	44.50	16.50
44.50	19.00	40.50	19.00
40.50	15.50	34.50	15.50
34.50	14.50	32.50	14.50
32.50	16.50	30.50	16.50
30.50	15.50	28.50	15.50
28.50	14.00	20.50	14.00

ORD 8 7 6 5 4 3 2 1

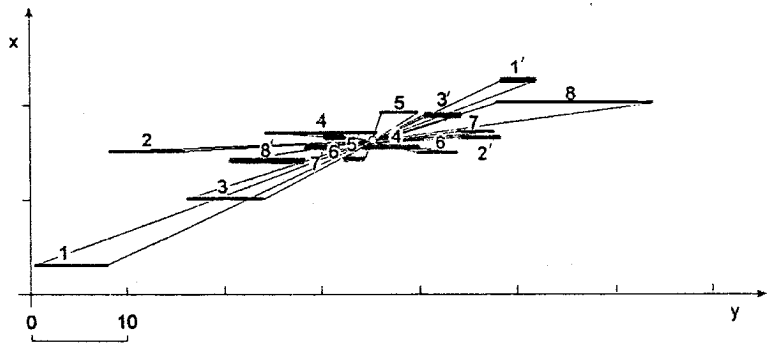


Fig. 3.72a

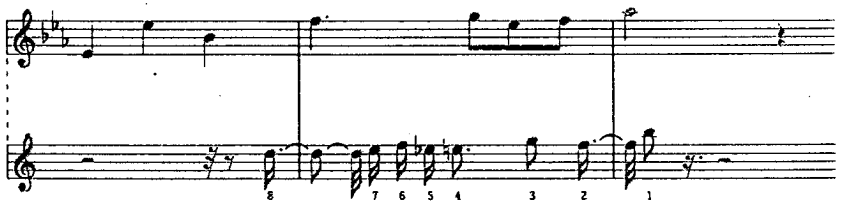



Fig. 3.72b

$k = -1$

70.00	29.00	62.00	29.00
62.00	17.00	54.00	17.00
54.00	22.00	46.00	22.00
46.00	15.00	34.00	15.00
34.00	13.00	30.00	13.00
30.00	17.00	26.00	17.00
26.00	15.00	22.00	15.00
22.00	12.00	6.00	12.00

ORD 8 7 6 5 4 3 2 1

Fig. 3.73a

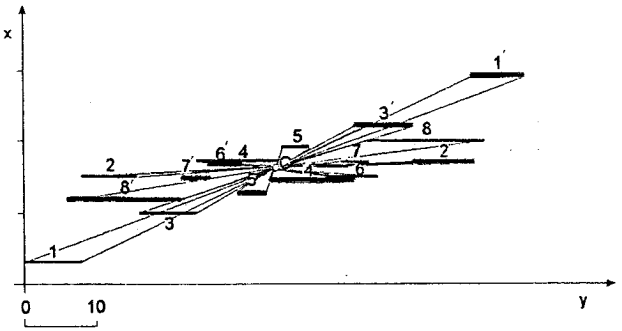
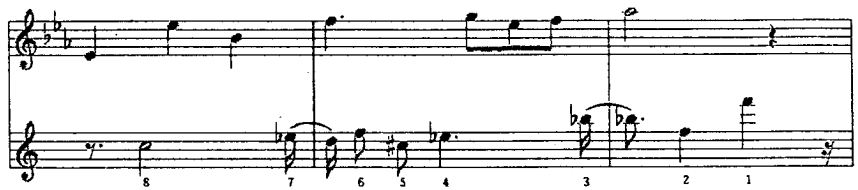


Fig. 3.73b

$k = -2$

105.00	42.00	89.00	42.00
89.00	18.00	73.00	18.00
73.00	28.00	57.00	28.00
57.00	14.00	33.00	14.00
33.00	10.00	25.00	10.00
25.00	18.00	17.00	18.00
17.00	14.00	9.00	14.00
9.00	8.00	–23.00	8.00

ORD 8 7 6 5 4 3 2 1

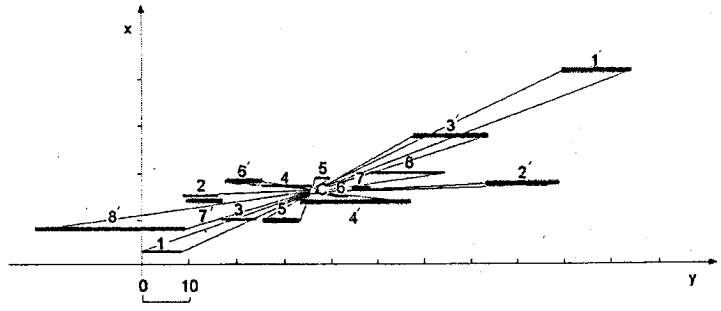


Fig. 3.74a

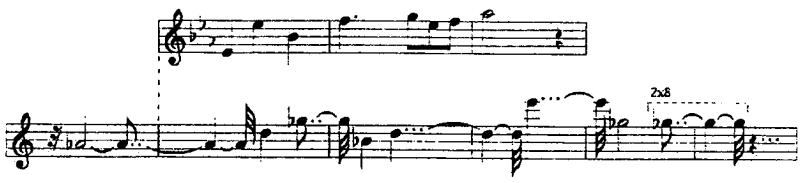



Fig. 3.74b

C (25, 29)

$k = 1.7$

–17.50	–15.20	–3.90	–15.20
–3.90	5.20	9.70	5.20
9.70	–3.30	23.30	–3.30
23.30	8.60	43.70	8.60
43.70	12.00	50.50	12.00
50.50	5.20	57.30	5.20
57.30	8.60	64.10	8.60
64.10	13.70	91.30	13.70

ORD 1 2 3 4 5 6 7 8

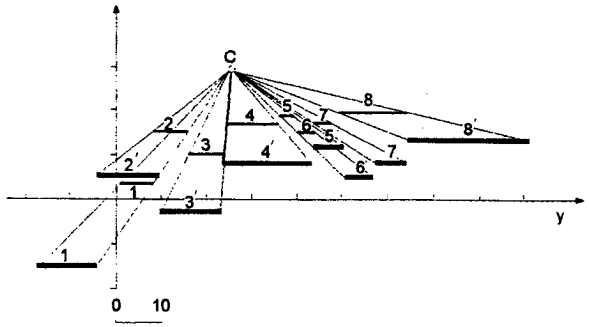


Fig. 3.75a

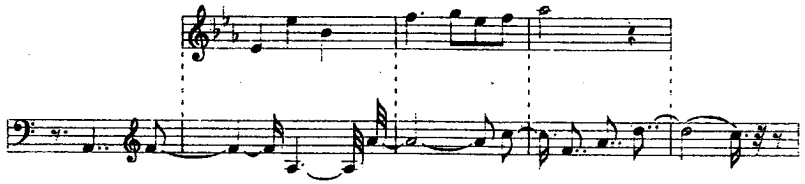



Fig. 3.75b

$k = 0.5$

12.50	16.00	16.50	16.00
16.50	22.00	20.50	22.00
20.50	19.50	24.50	19.50
24.50	23.00	30.50	23.00
30.50	24.00	32.50	24.00
32.50	22.00	34.50	22.00
34.50	23.00	36.50	23.00
36.50	24.50	44.50	24.50

ORD 1 2 3 4 5 6 7 8

Fig. 3.76a

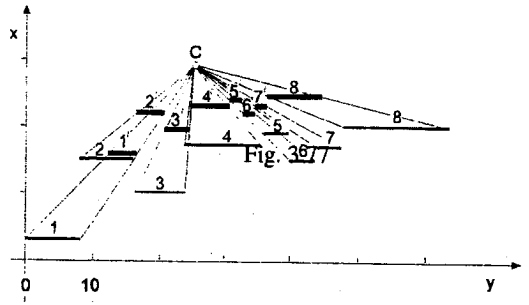
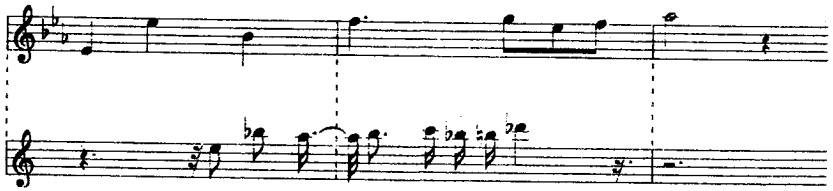


Fig. 3.76b

$k = -0.5$

37.50	42.00	33.50	42.00
33.50	36.00	29.50	36.00
29.50	38.50	25.50	38.50
25.50	35.00	19.50	35.00
19.50	34.00	17.50	34.00
17.50	36.00	15.50	36.00
15.50	35.00	13.50	35.00
13.50	33.50	5.50	33.50

ORD 8 7 6 5 4 3 2 1

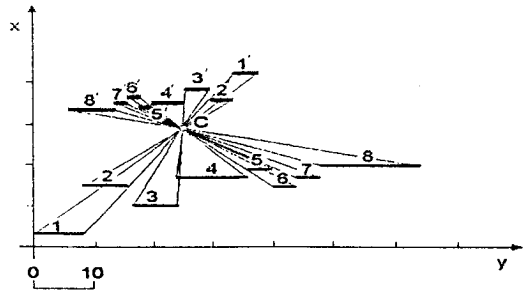


Fig. 3.77

$k = -1$

50.00	55.00	42.00	55.00
42.00	43.00	34.00	43.00
34.00	48.00	26.00	48.00
26.00	41.00	14.00	41.00
14.00	39.00	10.00	39.00
10.00	43.00	6.00	43.00
6.00	41.00	2.00	41.00
2.00	38.00	–14.00	38.00

ORD 8 7 6 5 4 3 2 1

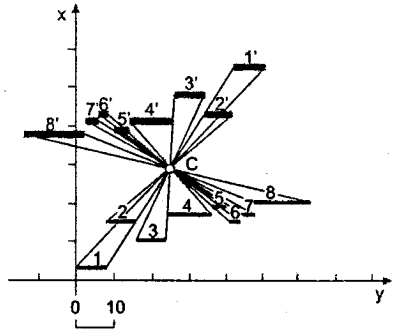


Fig. 3.78a

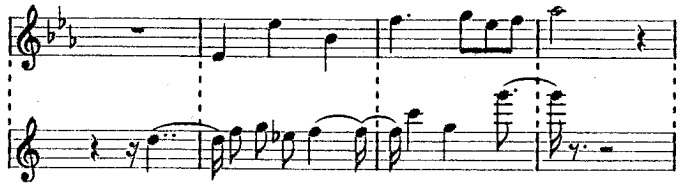



Fig. 3.78b

$k = -2$

75.00	81.00	59.00	81.00
59.00	57.00	43.00	57.00
43.00	67.00	27.00	67.00
27.00	53.00	3.00	53.00
3.00	49.00	–5.00	49.00
–5.00	57.00	–13.00	57.00
–13.00	53.00	–21.00	53.00
–21.00	47.00	–53.00	47.00

ORD 8 7 6 5 4 3 2 1

Fig. 3.79a

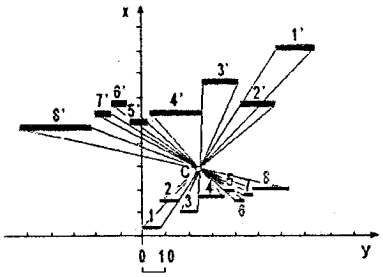
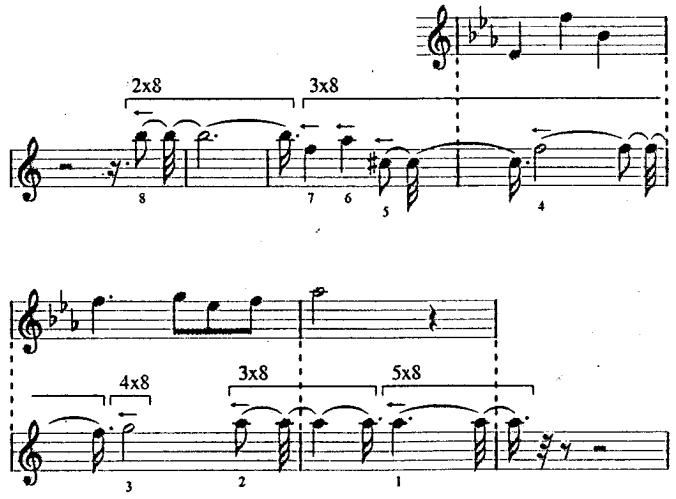


Fig. 3.79b

3.4. BEHAVIOUR OF SOUND FORMS UNDER THE ACTION OF TRANSFORMATIONS THROUGH ROTATION

We suggest the reader re-examine A.2 and A.3.16.

3.4.1. In order to facilitate the understanding of the changes a complex sound form undergoes when acted upon by rotation, let us survey first the behaviour of a sound *S* when submited to this transformation in the first two quadrants of the trigonometrical circle, depending on α angle (Fig. 3.80a, b).

$0^\circ < \alpha < \pi / 2$

0	12	8	12
8	20	20	20
32	2	36	2
36	15	40	15
40	–1	52	–1

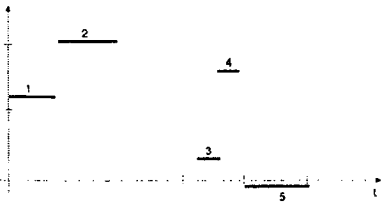


Fig. 3.80a

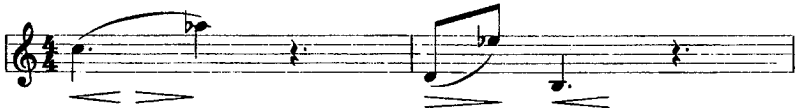
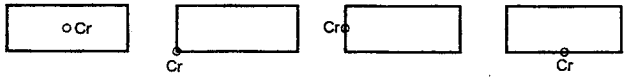



Fig. 3.80b

3.4.2. Study of the behaviour of sound form *F*, defined by the matrix subjected to transformation through rotation

As working method we will consider again the rectangle defining the sound form space in $\mathbf{R}^2$ (1.1.5) and depending on the position of the rotation center *Cr* position within this space, we will consider the following four cases:

1. *Cr* situated in the center of the rectangle ;
2. *Cr* situated in the lower left corner;
3. *Cr* situated in the middle of the left side;
4. *Cr* situated in the middle of the lower side.



For each case we will operate with a number of values for the angle of rotation α. Since (2.5.1a), except for cases when $\alpha = 0^\circ$, 90°, 180° and 270°, sounds of constant pitch transform into sounds of "glissando" type, at the end, we will also consider the compound transformation *OL * AO * RT*, where *RT* is a transformation through rotation and *OL* and *AO* the "ordinate lining up", and "abscissae ordering", respectively, allowing so to transform glissando sounds into constant pitch sounds (*OL*) and the sounds overlapping on certain sections into consecutive sounds (*AO*) (3.1.3; 3.1.4).

The sounds of the initial form are also provided with intensity notation, i.e. with crescendo (<) and diminuendo (>); subjected to rotation, the vector representing the sound may change its sense, the sound disappearance moment becoming the concurrence moment and conversely (A.3.16; 2.1.2; 2.5.1). As a result (in some further studied cases), the sounds having increasing and decreasing values of the intensity, will have the increasing and decreasing sense of these values inversed through rotation (Fig. 3.83).

1. (C_r in the center of the rectangle (3.4.1))
$(x, y)\ C_r\ (26, 9.5)$
$\alpha = 60^\circ$.

10.83	–11.77	14.83	–4.84
7.91	–0.84	13.91	9.55
35.50	10.95	37.50	14.41
26.24	20.91	28.24	24.37
42.09	16.37	48.09	26.77

ORD 2 4 3 5
1

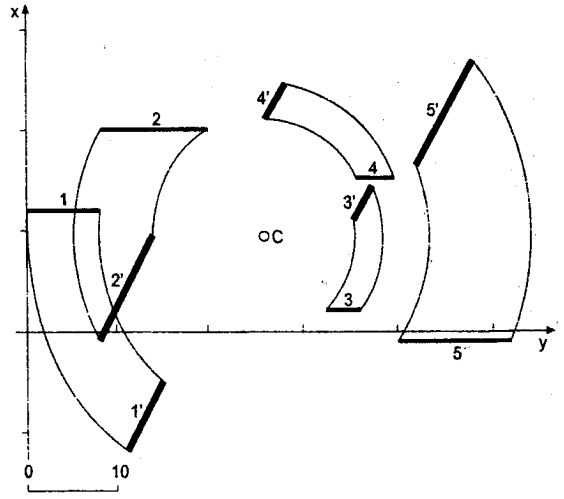


Fig. 3.81a

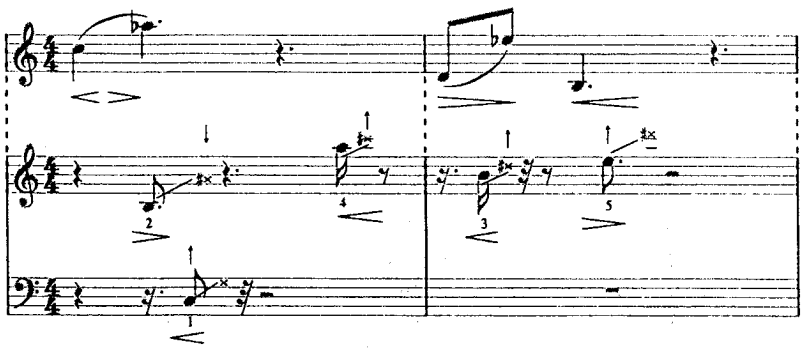


Fig. 3.81b

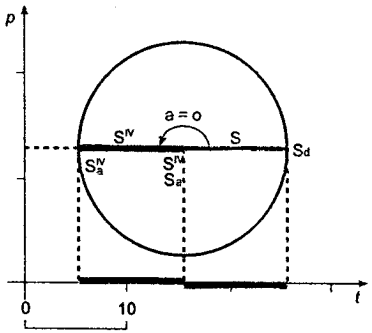


Fig. 3.82

Fig. 3.83

As with homothety and homology transformations, the sound concurrence order of a sound form may be changed through rotation. In order to be able to follow up the order transformation, each sound in the transformed image bears, in musical notation, a figure indicating its place in the initial image. In the following examples, we have marked with ORD the sound order; e.g. by ORD 24351 we mean that the order of sounds in the initial image (1, 2, 3, 4, 5) has become through rotation (2, 4, 5, 3, 1), i.e. the first sound in the initial image is the second one in the transformed image, the second – the fourth, a.s.o. This situation may be also traced on the geometric representation, where both images are numbered; ORD represents a permutation of 5 elements, $P_5 = 5!$

In the following we will consider the four cases above with respect to the values assigned to the rotation angle.

$\alpha = 135^\circ$

42.62	–10.65	36.96	–5.00
31.30	–10.65	22.82	–2.17
37.06	19.04	24.23	21.87
15.04	12.68	12.21	15.51
23.53	26.82	15.04	35.30

ORD 4 5 3
2 1

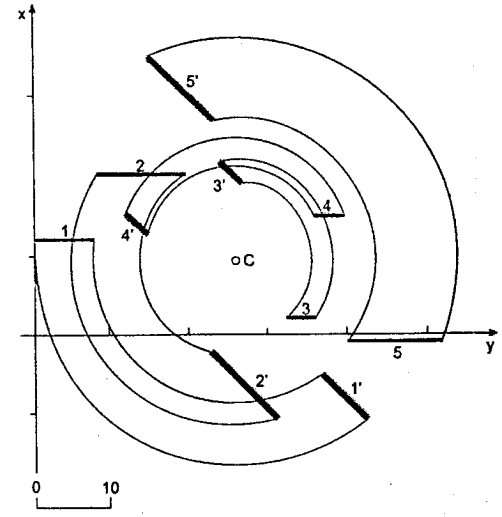


Fig. 3.84a

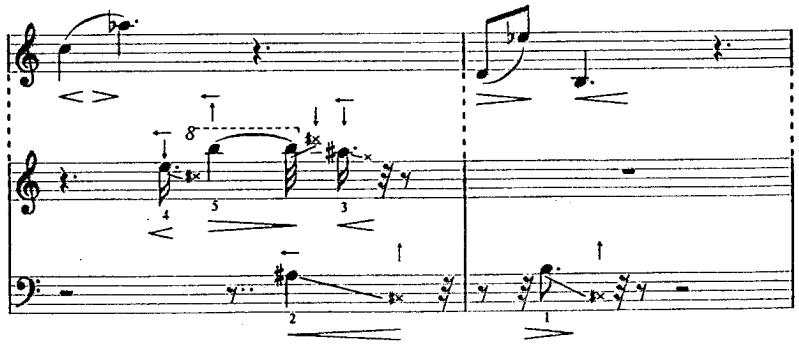


Fig. 3.84b

$\alpha = 180°$

52.00	7.00	44.00	7.00
44.00	–1.00	32.00	–1.00
20.00	17.00	16.00	17.00
16.00	4.00	12.00	4.00
12.00	20.00	0.00	20.00

ORD 5 4 3 2 1

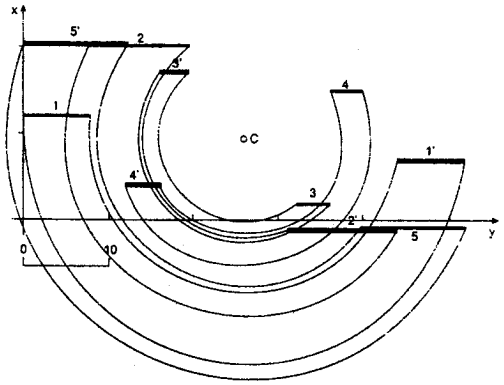


Fig. 3.85a

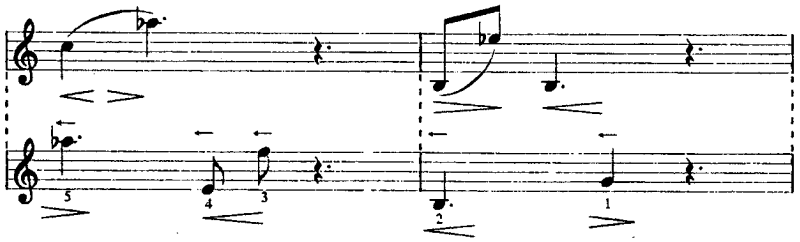


Fig. 3.85b

$\alpha = 225°$

46.15	26.12	40.50	20.46
46.15	14.80	37.67	6.31
16.45	10.56	13.63	7.73
22.82	–1.46	19.99	–4.29
8.68	7.03	0.19	1.46

ORD 5 3 4 2 1

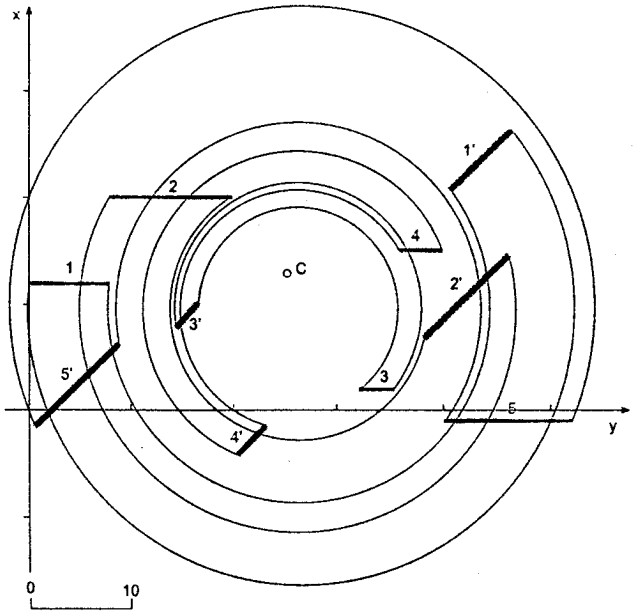


Fig. 3.86a

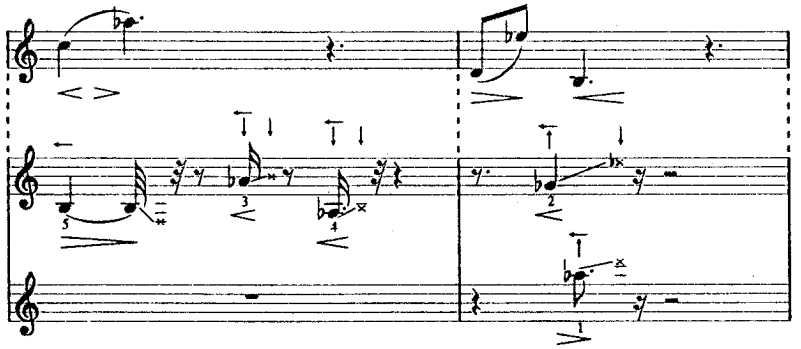


Fig. 3.86b

$\alpha = 270°$

28.50	35.50	28.50	27.50
36.50	7.50	36.50	15.50
18.50	3.50	18.50	–0.50
31.50	–0.50	31.50	–4.50
15.50	–4.50	15.50	–16.50

ORD 5 3 1 4 2

Fig. 3.87a

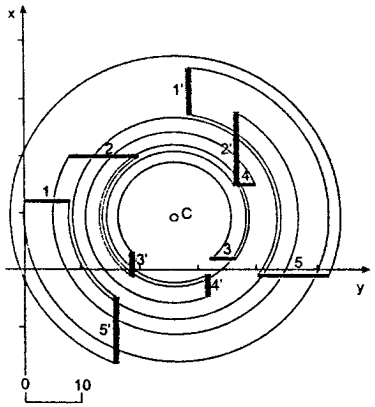
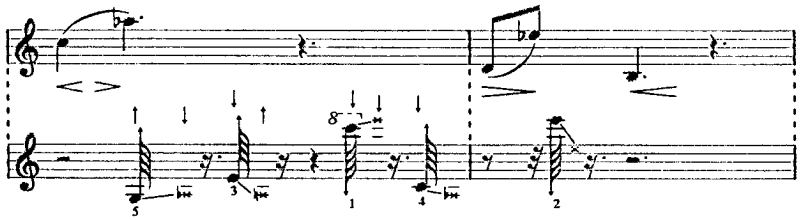


Fig. 3.87b



2. (C_T in the lower left corner)
(x, y) C_T (0, –1

$\alpha = 30°$

–6.50	10.26	0.43	14.26
–3.58	21.19	6.82	27.19
26.21	17.60	29.58	19.60
23.18	30.86	26.64	32.86
34.64	19.00	45.03	25.00

ORD 1 2 4 3 5

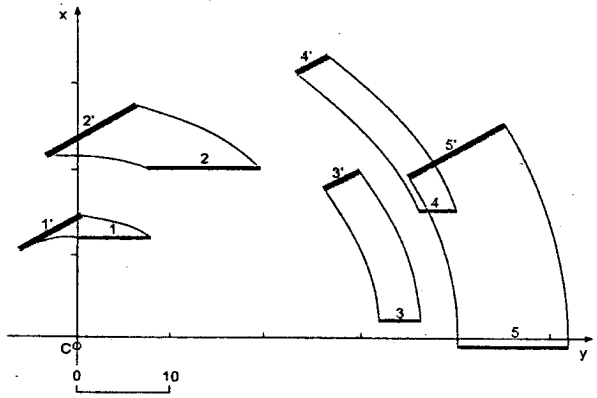


Fig. 3.88a

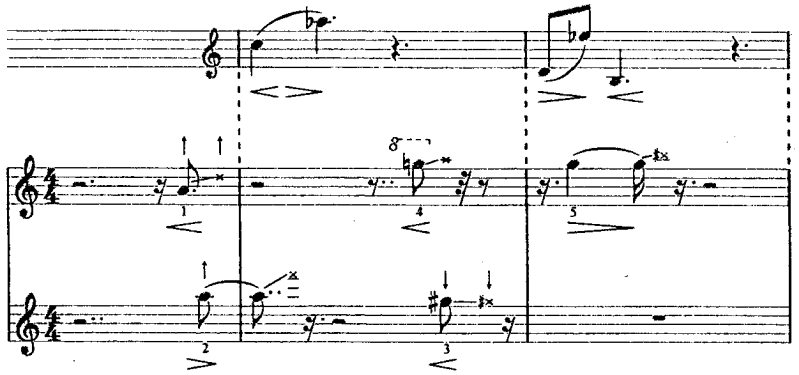


Fig. 3.88b

$\alpha = 45°$

–9.19	8.19	–3.54	13.85
–9.19	19.50	–0.70	28.00
20.51	23.75	23.33	26.58
14.14	35.77	16.97	38.60
28.28	27.28	36.77	35.77

ORD 1 4 3 5

2

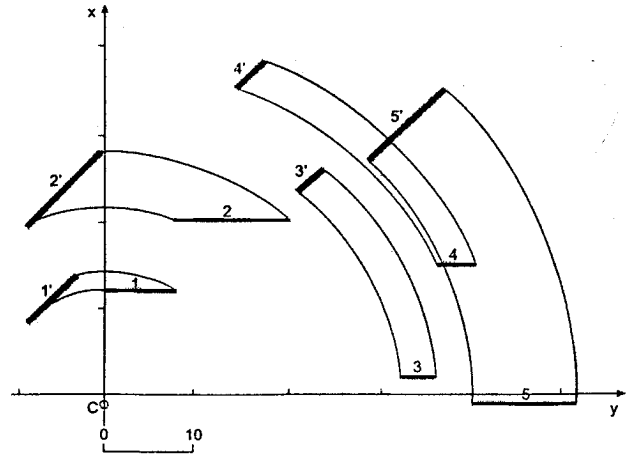


Fig. 3.89a

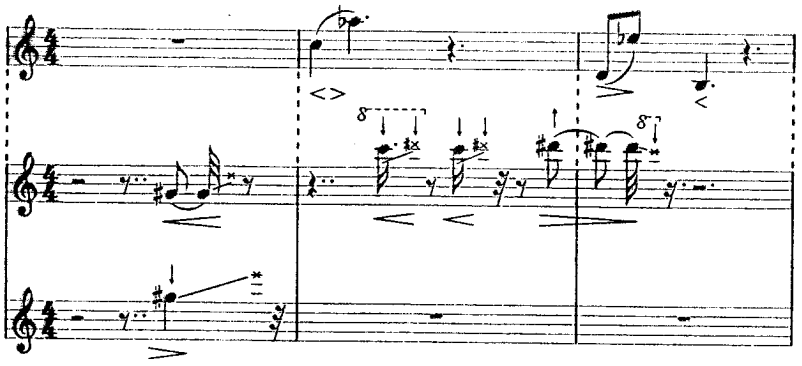


Fig. 3.89b

$\alpha = 90°$

–13.00	–1.00	–13.00	7.00
–21.00	–7.00	–21.00	19.00
–3.00	31.00	–3.00	35.00
–16.00	35.00	–16.00	39.00
0.00	39.00	0.00	51.00

ORD 2 4 1 3 5

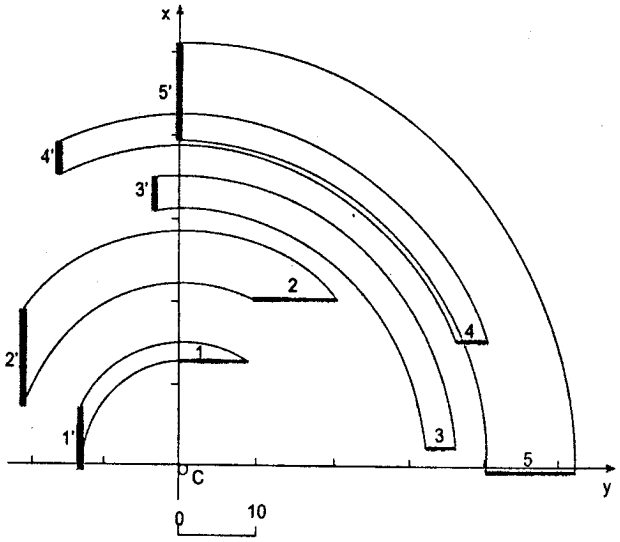


Fig. 3.90a

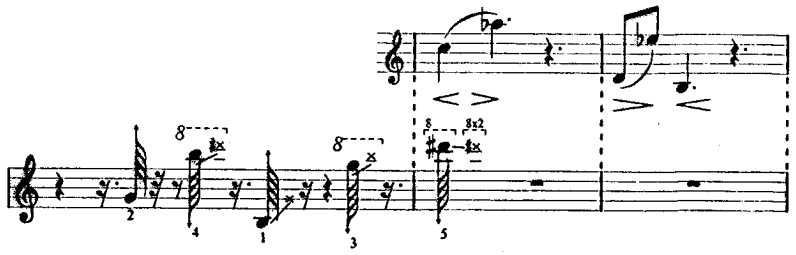


Fig. 3.90b

$\alpha = 135°$

–9.20	–10.19	–14.85	–4.54
–20.51	–10.19	–29.00	–1.70
–24.75	19.51	–27.58	22.34
–36.77	13.14	–39.60	15.97
–28.28	27.28	–36.77	35.77

ORD 4 5 2 3 1

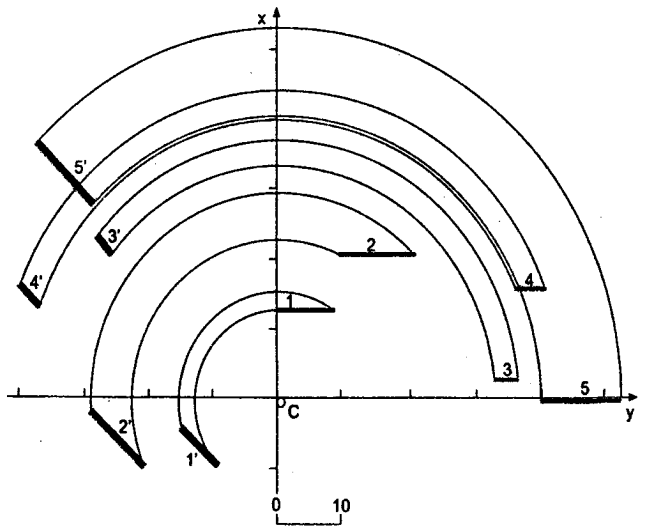


Fig. 3.91a

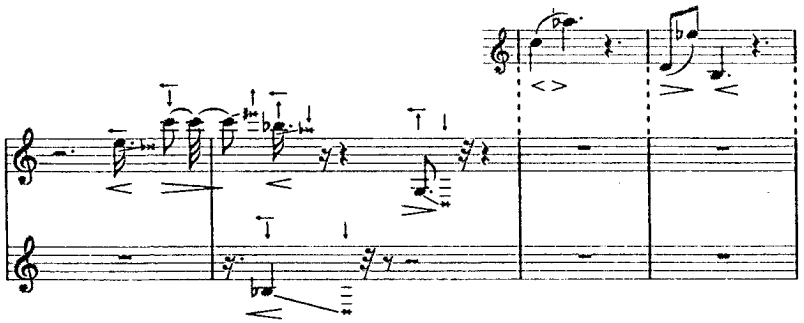


Fig. 3.91b

$\alpha = 180°$

0.00	−14.00	−8.00	−14.00
−8.00	−22.00	−20.00	−22.00
−32.00	−4.00	−36.00	−4.00
−36.00	−17.00	−40.00	−17.00
−40.00	−1.00	−52.00	−1.00

ORD 5 4 3 2 1

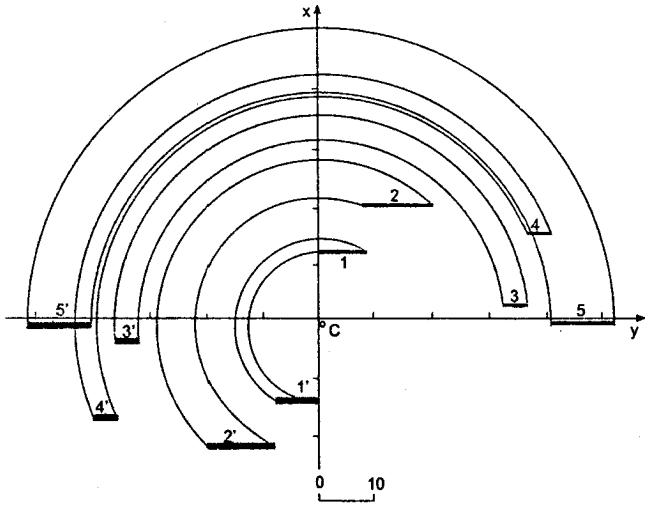


Fig. 3.92a

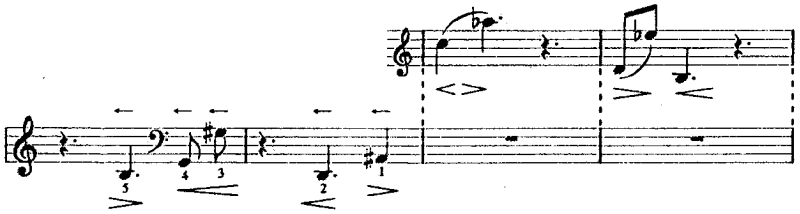


Fig. 3.92b

3. (C_r in the middle of the left side)

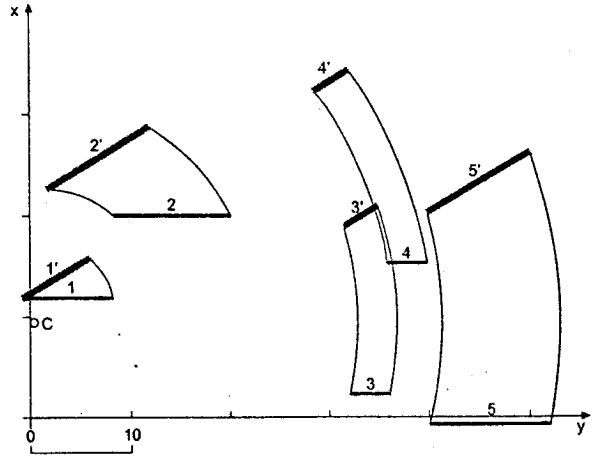
(x, y) C_r (0, 9.5)

$\alpha = 30°$

−1.25	11.67	5.68	15.67
1.68	22.59	12.07	28.59
31.46	19.00	34.93	21.00
28.43	32.26	31.89	34.26
39.89	20.41	50.28	26.41

ORD 1 2 4 3 5

Fig. 3.93a



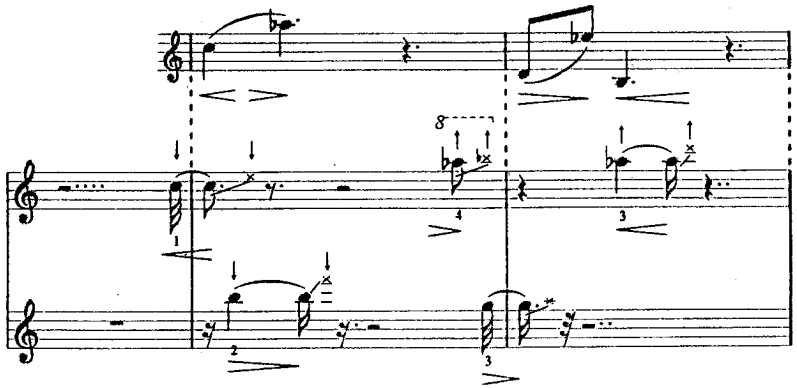


Fig. 3.93b

$\alpha = 180°$

0.00	7.00	–8.00	7.00
–8.00	–1.00	–20.00	–1.00
–32.00	17.00	–36.00	17.00
–36.00	4.00	–40.00	4.00
–40.00	20.00	–52.00	20.00

ORD 5 4 3 2 1

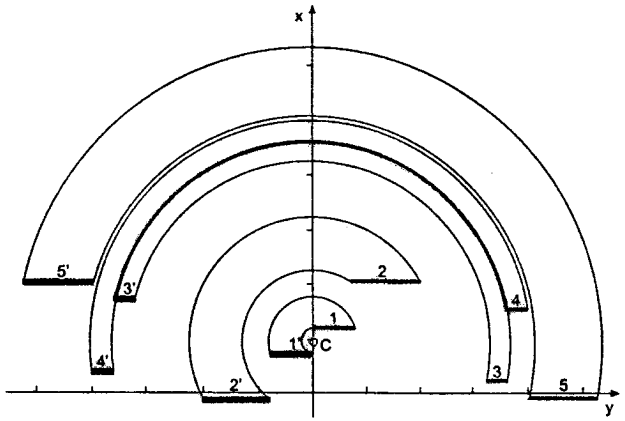


Fig. 3.94a

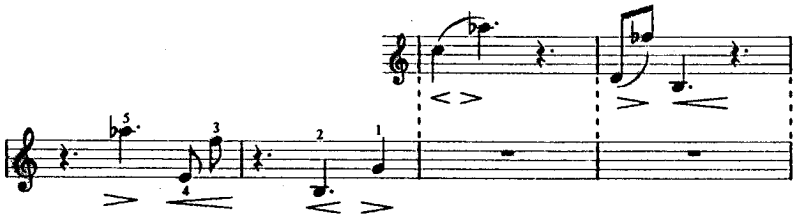



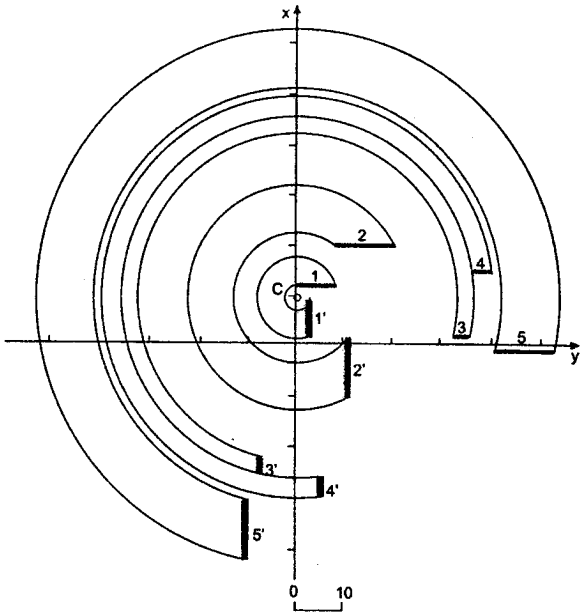
Fig. 3.94b

$\alpha = 270°$

2.50	9.50	2.50	1.50
10.50	1.50	10.50	–10.50
–7.50	–22.50	–7.50	–26.50
5.50	–26.50	5.50	–30.50
–10.50	–30.50	–10.50	–42.50

ORD 5 3 1 4 2

Fig. 3.95a



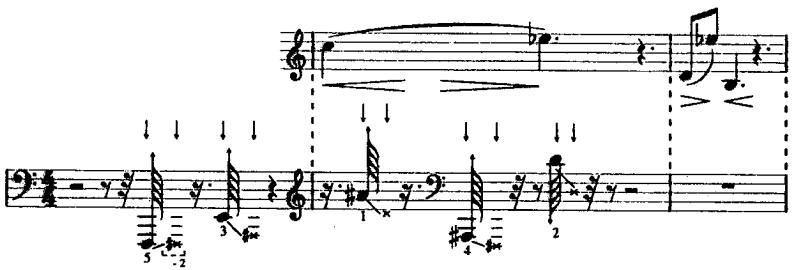


Fig. 3.95b

4. (C_r in the middle of the lower side)
(x, y) C_r (26, –1)

$\alpha = 30°$

–3.02	–2.74	3.91	1.25
–0.084	8.19	10.30	14.19
29.70	4.60	33.16	6.60
26.66	17.86	30.12	19.86
38.12	6.00	48.52	12.00

ORD 1 4 5
2 3

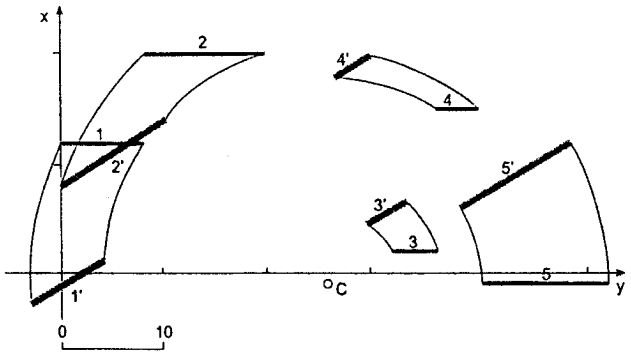


Fig. 3.96a

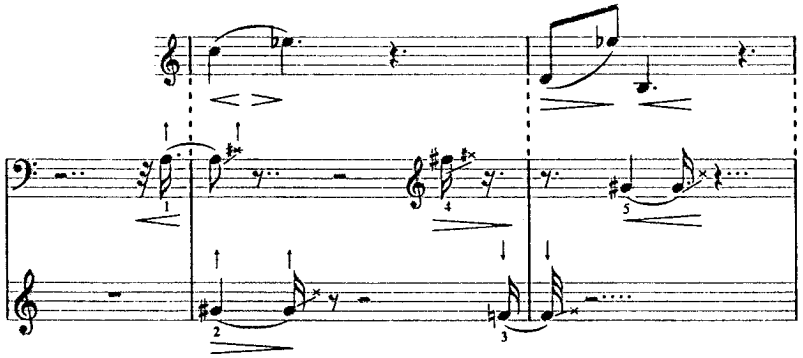


Fig. 3.96b

$\alpha = 180°$

52.00	–14.00	44.00	–14.00
44.00	–22.00	32.00	–22.00
20.00	–4.00	16.00	–4.00
16.00	–17.00	12.00	–17.00
12.00	–1.00	0.00	–1.00

ORD 5 4 3 2 1

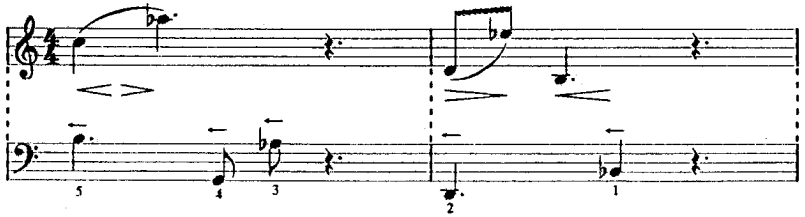
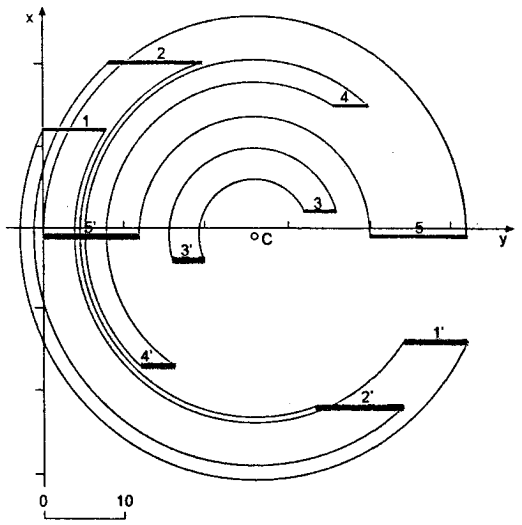


Fig. 3.97b

Fig. 3.97a



When rotation acts upon some sounds of constant pitch, represented in $\mathbf{R}^2$ by vectors parallel to the time axis, as is usually the case with musical sounds, depending on angle α, some *musical properties* of this transformation may be specified; the reader is invited to re-examine (2.5.1).

3.5. BEHAVIOUR OF SOUND FORMS UNDER THE ACTION OF TRANSFORMATIONS THROUGH INVERSION

We suggest the reader re-examine A.2 and A.3.17.

Taking into account the impossibility of representing accurately the results of transformations through inversion in musical notation, we will examine the behaviour of a sound form under inversion, limiting ourselves to the global, graphic aspect.

With the given examples, the same sound form *SF* is observed, as well as the same value of the *q* module. The relative position between *SF* and the inversion circle is variable.

1. The sound form *SF* is inside the inversion circle.

The sound transforms from *SF* are arcs of a circle, outside the inversion circle, with a longer duration than the initial sounds.

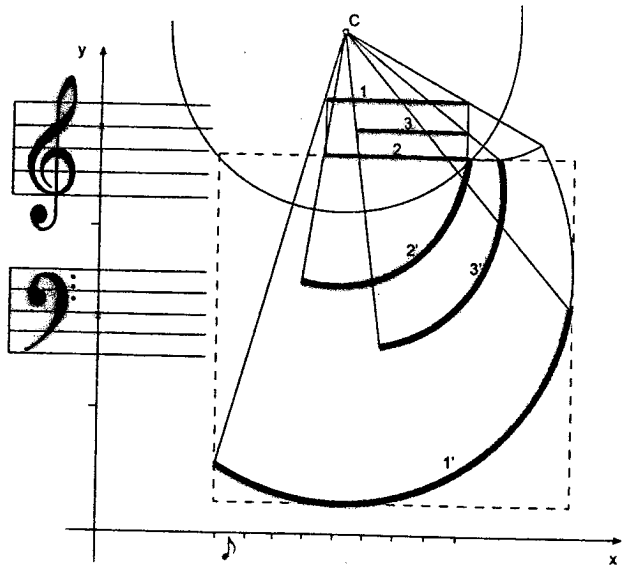


Fig. 3.98

For a transformation of *SF* by inversion with values having a minus sign, the obtained images are symmetrical with respect to the pole O (the center of the inversion center) with the images obtained by the corresponding positive values.

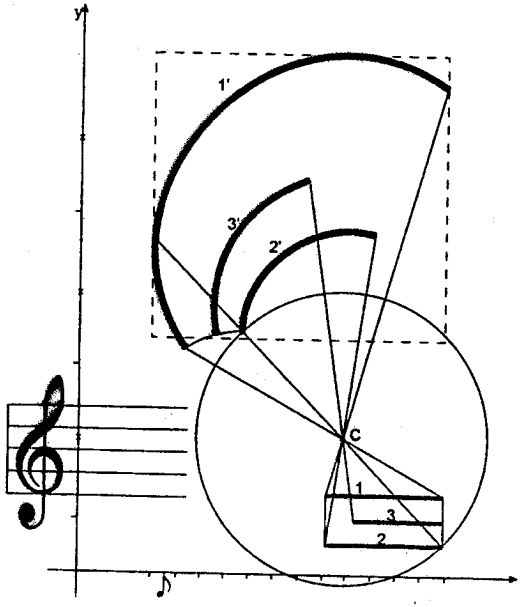


Fig. 3.99

2. The sound form *SF* intersects the inversion circle.

Inversion with + sign (Fig. 3.100)

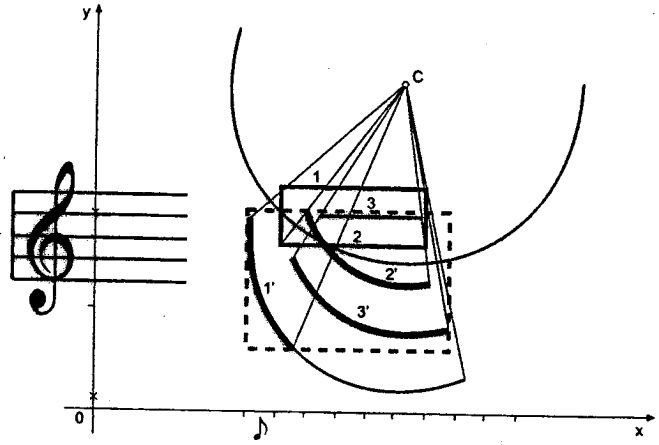
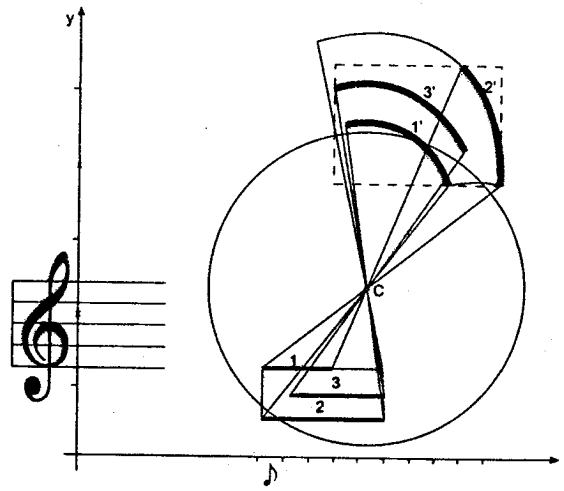


Fig. 3.100

Inversion with – sign (Fig. 3.101)

Fig. 3.101



3. The sound form *SF* is outside the inversion circle. The transforms of the sounds of *SF* have a shorter duration than the initial sounds.

Inversion with + sign (Fig. 3.102)

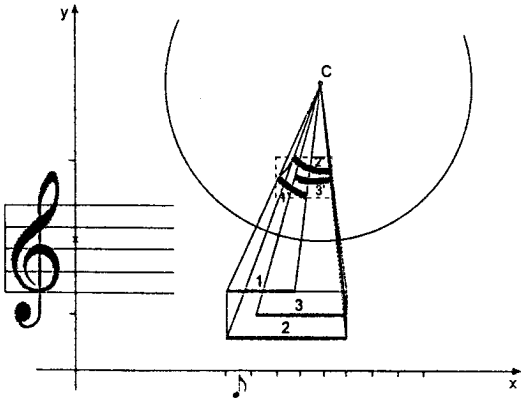
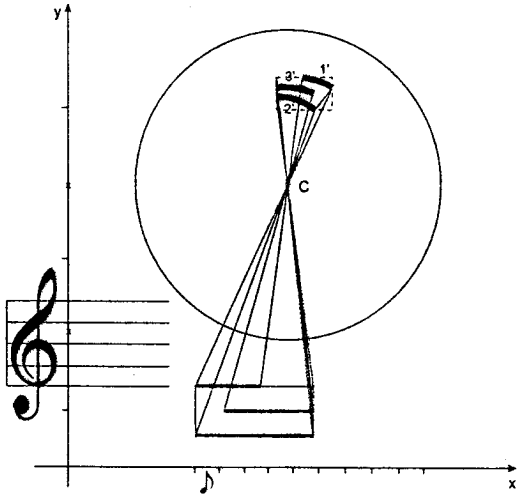


Fig. 3.102

Inversion with – sign (Fig. 3.103)

Fig. 3.103



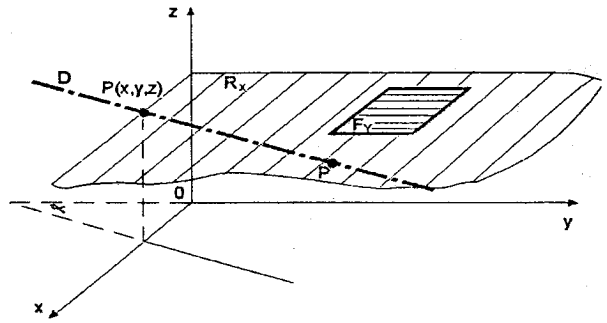
3.6. TRANSFORMATIONS IN SOUND SPACE R^3. BEHAVIOUR OF SOUND FORMS UNDER THE ACTION OF TRANSLATION, ROTATION AND PROJECTION. COMPOUND TRANSFORMATIONS

3.6.1. General remarks

In (1.1.5) we have used the concept of virtual sound space defined as a three-dimensional operational space, two of its dimensions being the real sound space dimensions (time and pitch), the third one (Oz) being an abstract algebraical dimension. We refer the reader to the chapter on the transformations in space $\mathbf{R}^3$ (A.2; A.3.19) which will enable him to "get" the visual aspect of such transformations, considering the rectangle represented in these examples as being the space of a virtual sound form (*Fv*).

In the following, we will examine the behaviour of a sound form Fv subjected to a compound transformation product: translation and rotation in $\mathbf{R}^3$ and projection from $\mathbf{R}^3$ in $\mathbf{R}^2$. The reader is invited to also re-examine (A.2). As indicated there, the virtual sound form *Fv* is obtained by the translation of a real sound form *Fv* from $\mathbf{R}^2$ in $\mathbf{R}^3$ with respect to an orthogonal vector on $\mathbf{R}^2$. The translation is carried out in a parallel plane with $\mathbf{R}^2$ at a "height" *z*. *Fv* rotates in $\mathbf{R}^3$ at an angle α around a rotation axis *A* defined by the angle β made by *A* with the *Ox* axis and a point *P* in $\mathbf{R}^3$. In order to help the reader to better "see" the rotation, in the following examples, *P* belongs to the sound space of *Fv*, as defined under (1.1.4), i.e. the space of the rectangle whose sides are parallel to the *xOy* axes and pass through points of abscissae and ordinates of minimum and maximum value belonging to the virtual sounds of *Fv* (Fig. 3.104).

Fig. 3.104



The following examples show the initial image represented by the sounds *Si* ($i = 1, 2, ..., n$) and the obtained image with the considered product of transformations, represented by the sounds *S'i*. The cylindrical orthogonal projection of the rotation axis *A* and of the projection center *Cp* in $\mathbf{R}^2$ are also represented. The point *P* is defined by the *xA*, *yA* coordinates (*zA* being, by definition, equal to *zFv*), since *F* is in the same parallel plane with $\mathbf{R}^2$ where *Fv* is located. The projection center is defined by the *xCp*, *yCp* and *zCp* coordinates. With the cylindrical projection, the angle π made by the projection lines with $\mathbf{R}^2$ should also be specified. In our examples, only the orthogonal cylindrical projection ($\pi = 90^\circ$) is considered.

3.6.2. Taper projection

Considering that the sound form is "raised" over the space $\mathbf{R}^2$ ($0 < zCp$), Cp may be situated "over" the sound form ($zCp > zFv$), between the sound form and $\mathbf{R}^2$ ($zF > zCp > 0$), or "under" $\mathbf{R}^2$ ($zCp < 0$) (A.4.1).

Depending on the position of the axis of rotation, on the rotation angle of the form around the axis and on the position of the center of projection, a practically infinite set of transformations of the initial form may be obtained.

For each of the above-mentioned cases, we will give examples accompanied, if need be, by some remarks. In order to easier interpret the transformation process, we will also have some images in $\mathbf{R}^3$. Also in the following examples, the matrix defining the virtual form after rotation, the coordinates of the point and value of the angle defining the axis of rotation, as well as the value of the angle of rotation of the virtual form around the axis, are given.

3.6.3. Case 1 ($zCp > zFp$)

General rule: the F image in $\mathbf{R}^2$ (F') is dilated with respect to the image obtained through orthogonal center projection (A.3.15).

N.B. In (A.3.15) we have indicated that, from the geometric point of view, in the above-mentioned case, the form dilates. Now, it is necessary to show that, from the musical point of view, depending on the position of the center of projection, of the axis of rotation and the angle of rotation of Fv around the axis, situations might occur, when, although through projection, the distance between the ends of some segments representing sounds might increase, the duration of the respective sounds might be diminished on account of the angle made by the transformed segment with the time axis: the projection on the time axis of the transformed segment may define smaller time intervals than the projection of the initial segment. An identical remark may be made regarding the changing of pitch interval values between two sounds (Fig. 3.105a,b).

$zFv = 100$, $\alpha = 90°$, $(x, y)P = 0, -1$, $\beta = 180°$, $(x, y, z)Cp = 26, 9.5, 200$

Cp is "over" the center of Fv at the "height" $z = 200$. An inversion of the sound order and a form expansion are obtained. The sense of intensity signs is inversed.

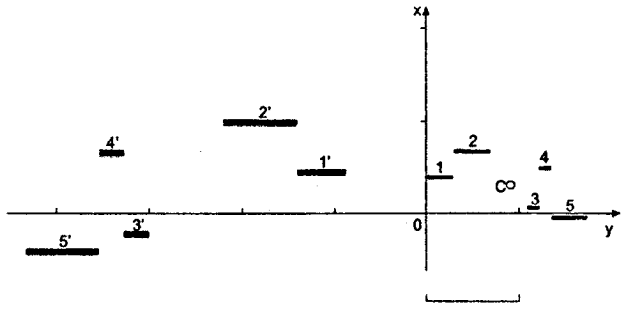


Fig. 3.105a

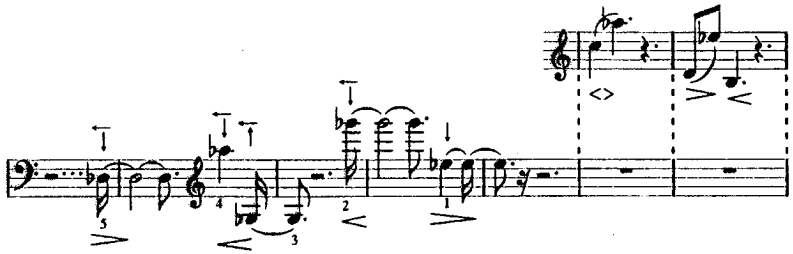


Fig. 3.105b

Inversion and expansion

$zFv = 120$, $\alpha = 30°$, $(x, y, z)P = 14, 10$, $\beta = 45°$, $(x,y,z)Cp = 14, 10, 250$

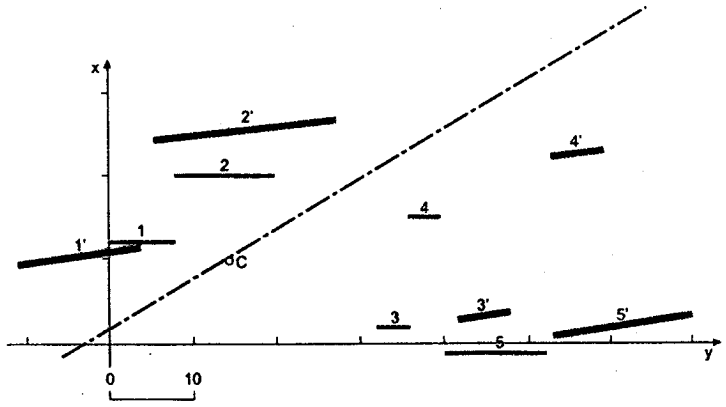


Fig. 3.106a

$x(Cp)$ coincides with $x(P)$. The sounds S'_1 and S'_2 and, respectively, S'_3 and S'_4 are no longer concatenated. S'_5 anticipates S'_4 which overlaps S'_5.

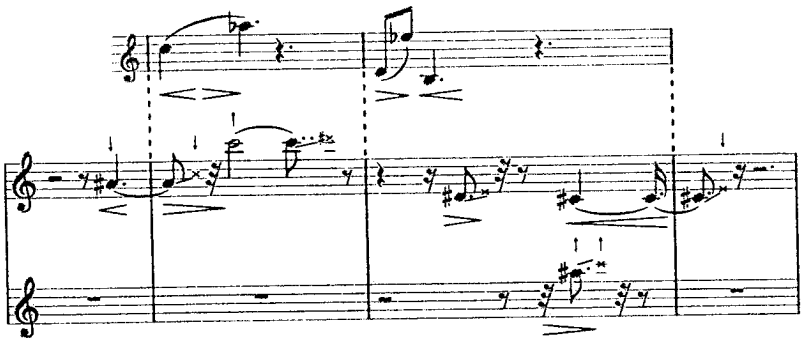



Fig. 3.106b

$zFv = 100$, $\alpha = 100°$, $(x, y,z)P = 0, 6$, $\beta = 145°$, $(x\ y\ z)Cp = 2, 40, 200$

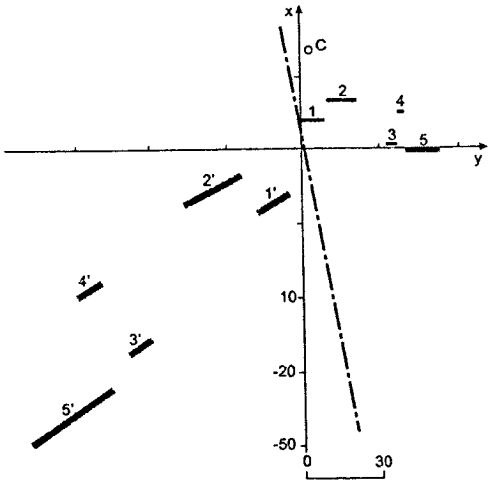


Fig. 3.107a

The sound order and sense is inverted. S'_4 overlaps S'_5. S'_5, S'_4 and S'_3 are imaginary sounds (1.1.3).

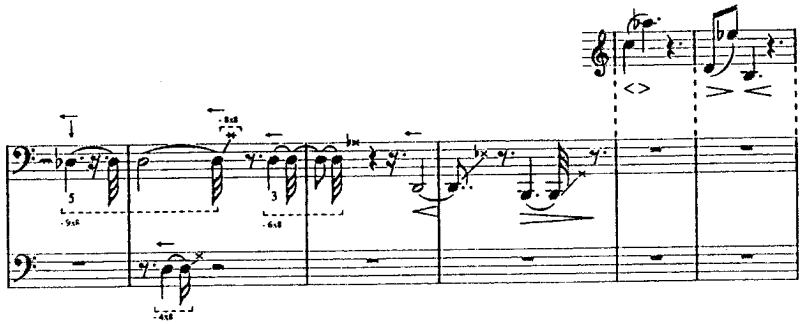



Fig. 3.107b

$zFv = 100$, $\alpha = 0°$, $(x, y)P = 0, 0$, $\beta = 45°$, $(x, y,z)Cp = 0, 0, 200$

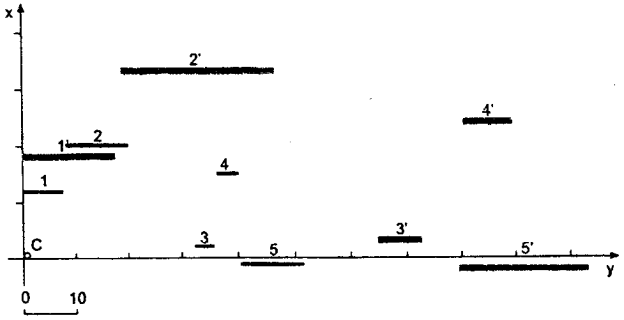


Fig. 3.108a

$x(P)$ and $(x\ )Cp$ coincide in the origin of axes. Axis A is parallel to the Fv sounds ($\beta = 0°$). By transformation, the sounds image remains parallel to the axis. Hence, the sounds remain at constant pitch.

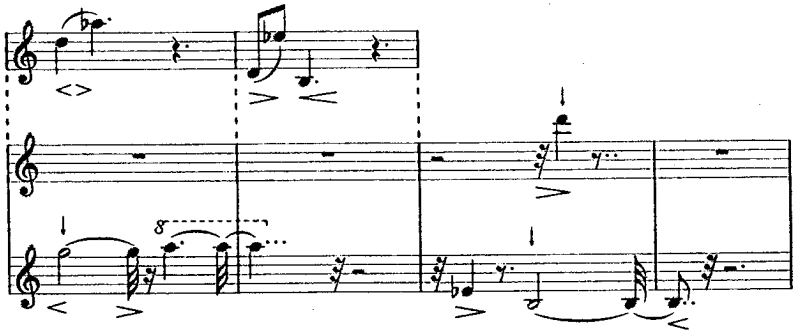


Fig. 3.108b

$zFv = 75$, $\alpha = 0°$, $(x, y)P = 26, 9.5$, $\beta = 135°$, $(x,y,z)Cp = 26, 9.5, 150$

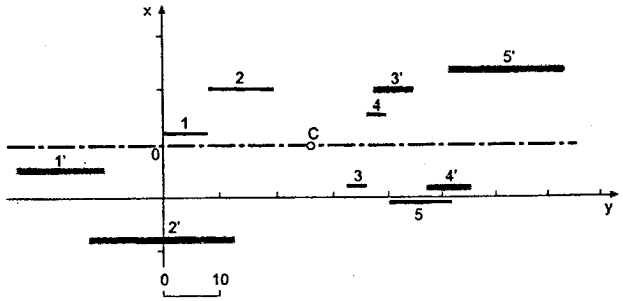


Fig. 3.109a

x(P) and *x(Cp)* coincide, *A* crosses the *Fv* center. Since $\beta > 90°$, we obtain a "reversion" combined with a dilation. $\alpha = 0°$: The *F'* sounds have constant pitch.

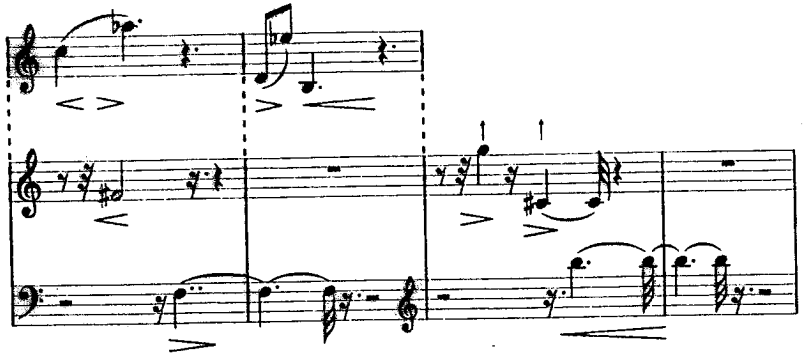


Fig. 3.109b

$zFv = 75$, $\alpha = 0°$, $(x, y)P = 26, 9.5$, $\beta = 135°$, $(x, y, z)Cp = 20, 30, 120$

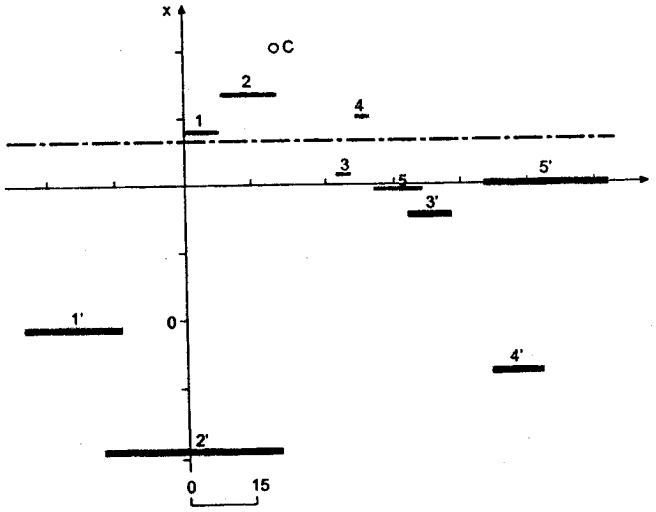


Fig. 3.110a

Reversion, dilation, S'_2, S'_4 are imaginary sounds!

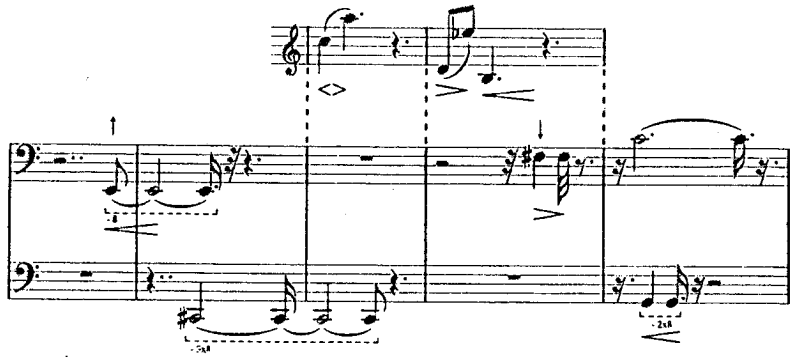


Fig. 3.110b

$zFv = 75$, $\alpha = 135°$, $(x, y)P = 0, -10$, $\beta = 90°$, $(x, y, z)Cp = 26, 9.5, 150$

S'_i are glissando sounds of great "slope" and shorter duration than those of the initial forms (2.3.1). S'_3, $S'4$ and S'_5 are imaginary sounds.

Fig. 3.111a

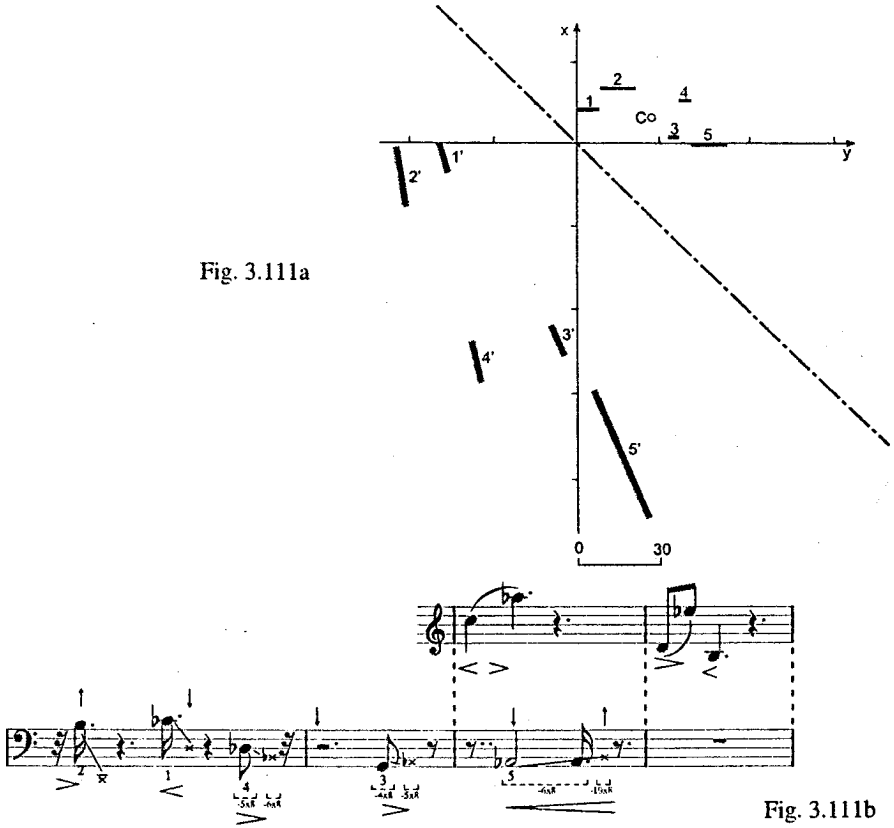


Fig. 3.111b

3.6.4. Case 2 ($o < zCp < zFv$)(Cp between Fv and R^2)

General rule: the image of Fv appears as inversed and reversed as against the image obtained through cylindrical and orthogonal projection. Particularly, when $zFv > Cp > zFv/2$, the image of F' is inversed, reversed and dilated; when $zCp = zFv/2$, the image is a central symmetry. The reader meets again the "reversion of inversion" in music.

When $0 < zCp < zFv/2$, the image of F' is inversed, reversed and compressed. The image is compressed as $zCp \rightarrow 0$.

When $zCp = 0$ (Cp is in the $\mathbf{R}^2$ space), the image of Fv is reduced to a point coinciding with Cp (A.4.1.d).

$zFv = 100$, $\alpha = 0°$, $(x,y)P = 0, -1$, $\beta = 150°$, $(x, y\ z)Cp = 26, 9.5, 50$

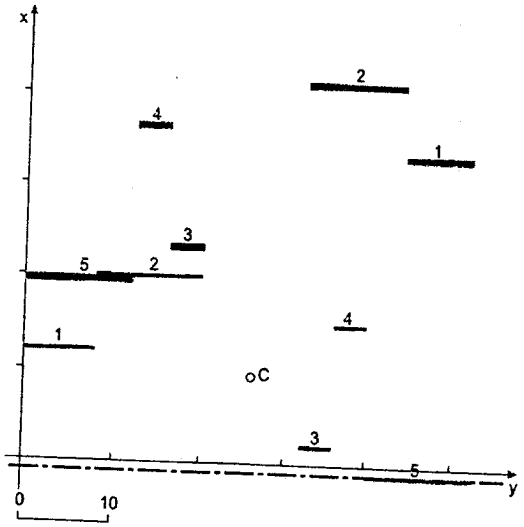


Fig. 3.112a

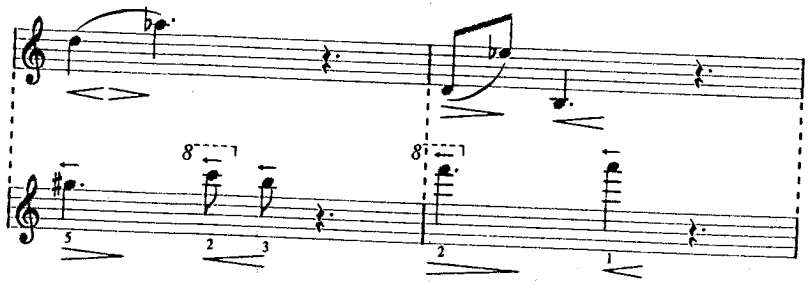


Fig. 3.112b

$zFv = 100$, $\alpha = 0°$, $(x, y)P = 0, -1$, $\beta = 0°$, $(x, y\ z)Cp = 26, 9.5, 25$

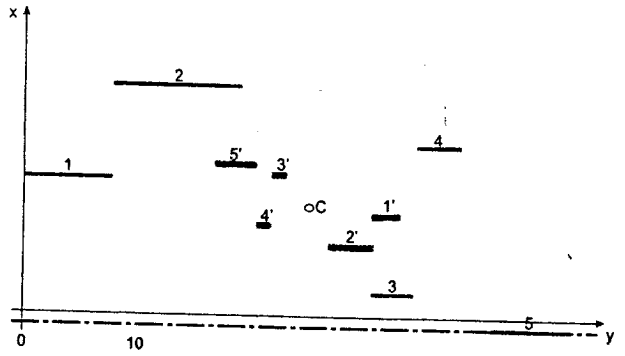


Fig. 3.113a

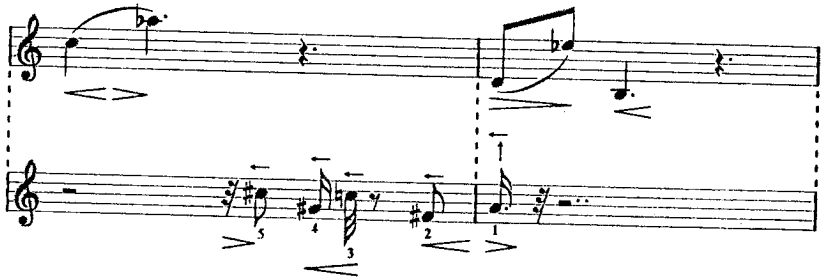


Fig. 3.113b

$zFv = 100$, $\alpha = 0°$, $(x, y, z)P = 0, -1$, $\beta = 0°$, $(x, y, z)Cp = 26, 9.5, 75$

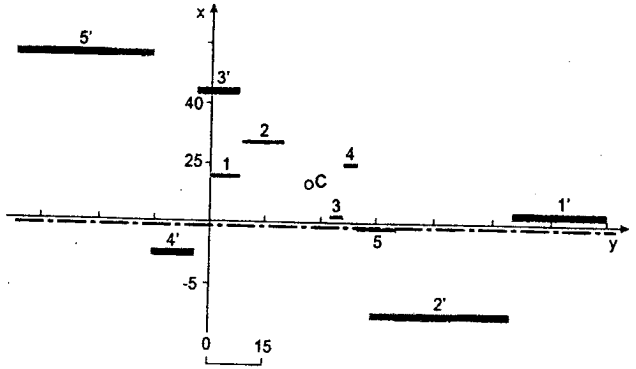


Fig. 3.114a

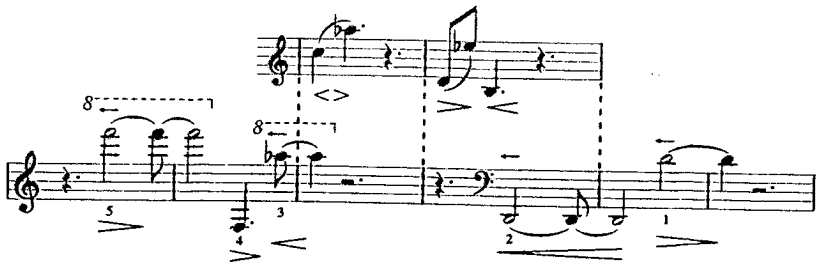


Fig. 3.114b

$zFv = 100, \alpha = 90°, (x, y,z)P = 0, -1, \beta = 180°, (x, y, z)C = 26, 9.5, 50$

Fig. 3.115a

Center symmetry! (Schoenberg)

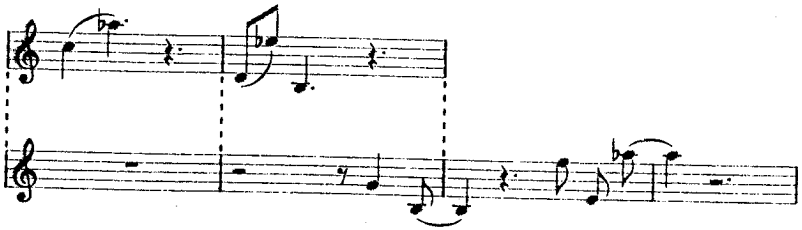



Fig. 3.115b

$zFv = 100, \alpha = 90°, (x, y,z)P = 0, -1, \beta = 250°, (x\, y, z)C = 26, 9.5, 50$

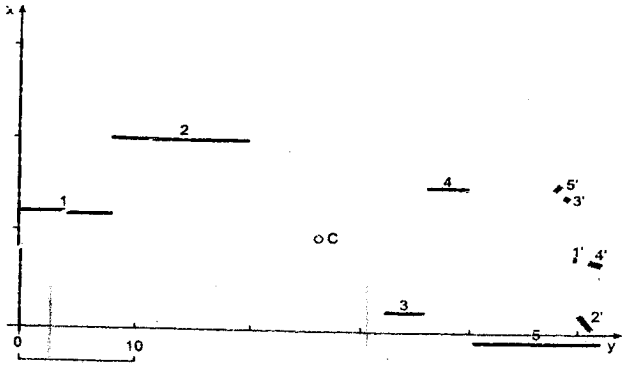


Fig. 3.116a

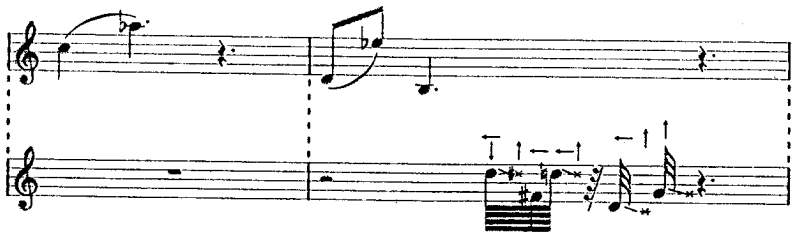


Fig. 3.116b

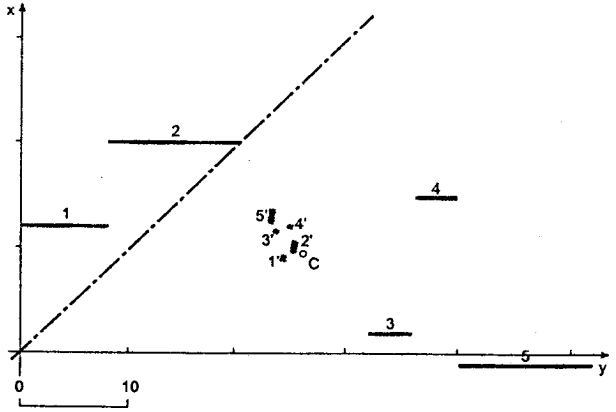
3.6.5. Case 3 ($zCp < 0$)(Cp situated "below" sound space R^2)

The image of Fv is "compressed" as against the image obtained through orthogonal cylindric projection (A.3.15). From its agglomeration in a point, when $zCp = 0$, the image of the Fv is "redilated" (in direct condition, without inversion and reversion, as in case 2) as Cp takes decreasing values of z, $z < 0$.

When $zCp \to -\infty$, the projection lines become cylindrical; Fv becomes congruent with F.

$zFv = 100, \alpha = 45°, (x, y)P = 0, 0, \beta = 140°, (x, y, z)Cp = 26, 9.5, -10$

Fig. 3.117a



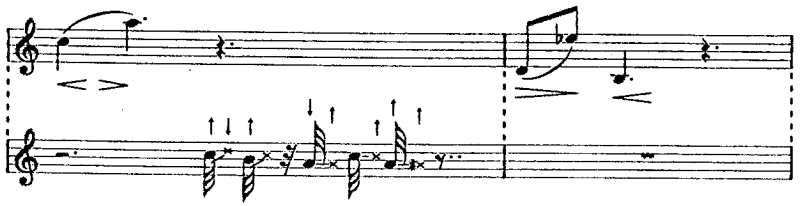


Fig. 3.117b

All the *F'* sounds are null sounds (duration zero).

$zFv = 100$, $\alpha = 0°$, $(x, y)P = 0, -1$, $\beta = 240°$, $(x, y, z)Cp = 20, 9.5, -20$

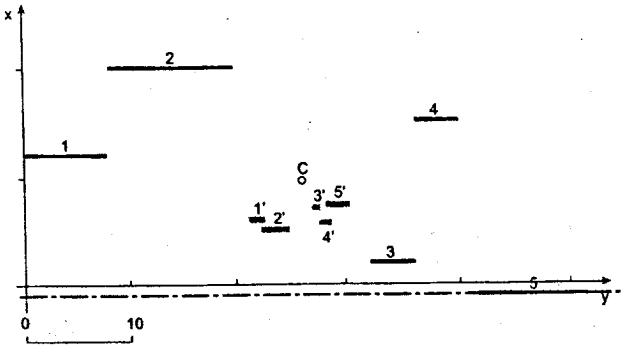


Fig. 3.118a

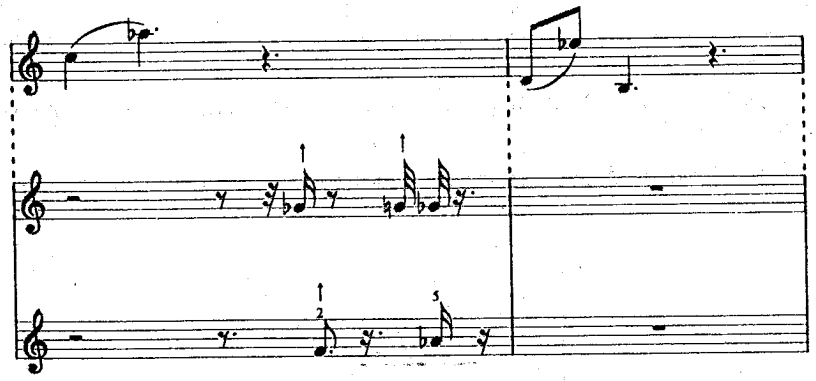



Fig. 3.118b

$zFv = 100$, $\alpha = 90°$, $(x, y)P = 0, -1$, $\beta = 140°$, $(x, y, z)Cp = 20, 9.5, -20$

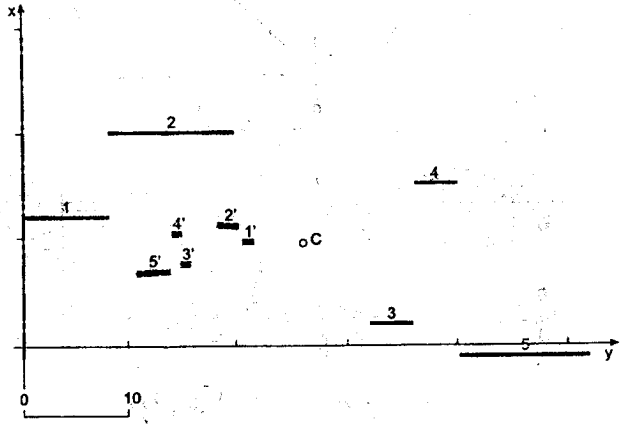


Fig. 3.119a

The 140° rotation moves the *Fv* sounds in the direction of the negative value abscissae. *Cp*, situated at $x = 26$, $y = 9.5$, again "absorbs" sounds to the positive abscissae zone.

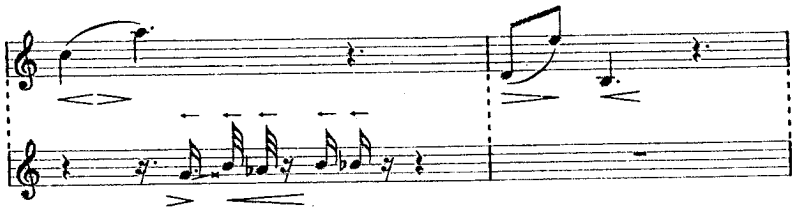



Fig. 3.119b

$zFv = 100$, $\alpha = 0°$, $(x, y)P = 26, 9.5$, $\beta = 180°$, $(x, y, z)Cp = 26, 9.5, -10$

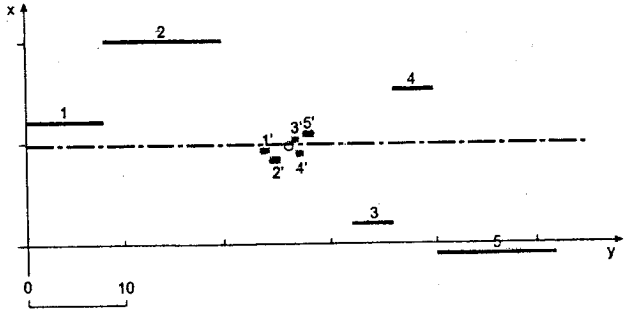


Fig. 3.120a

After a reversion around the axis, the sound form is "absorbed" towards the center, being projected in $\mathbf{R^2}$.

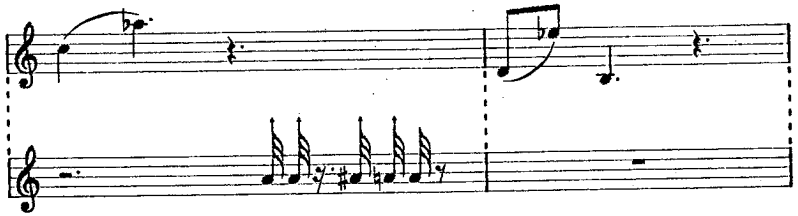


Fig. 3.120b

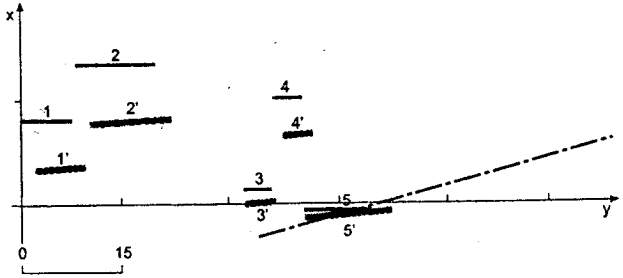


Fig. 3.121a

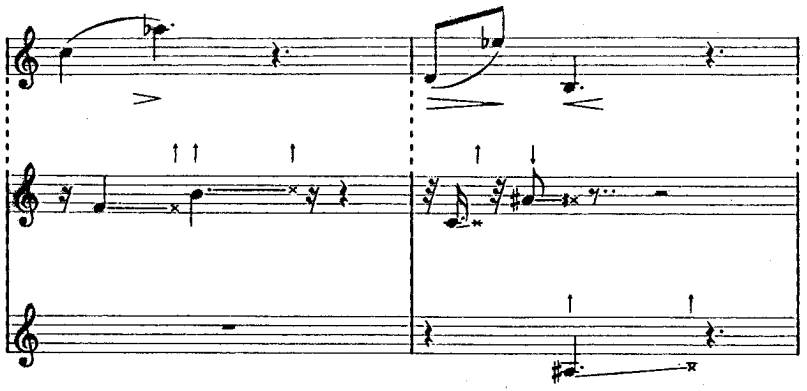


Fig. 3.121b

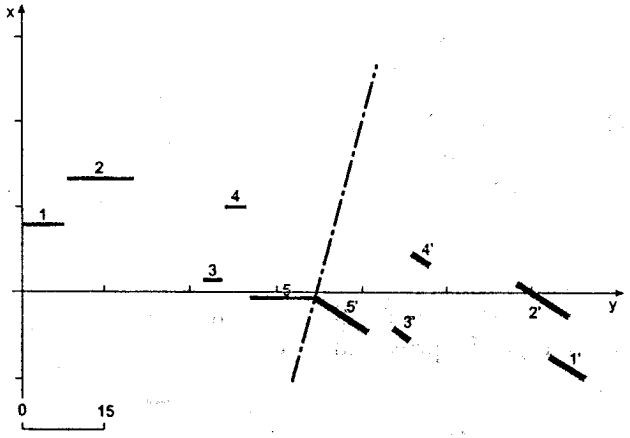


Fig. 3.122a

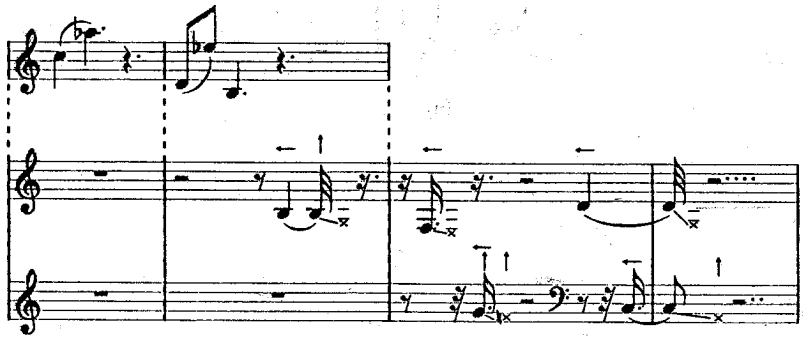


Fig. 3.122b

3.6.6. Case 4. Cylindrical projection ($zCp \to \infty$)

In the following we will give some examples of transformations through orthogonal cylindric projection. The reader will meet again the classical reversions and inversions in music.

$$zFv = 100,\ \alpha = 90°,\ (x, y)P = 0, -1,\ \beta = 30°$$

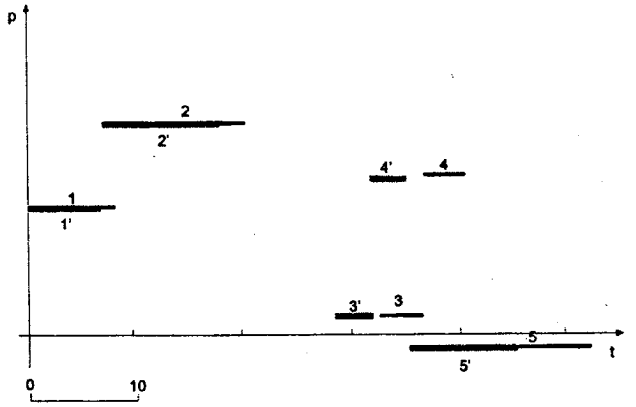


Fig. 3.123a

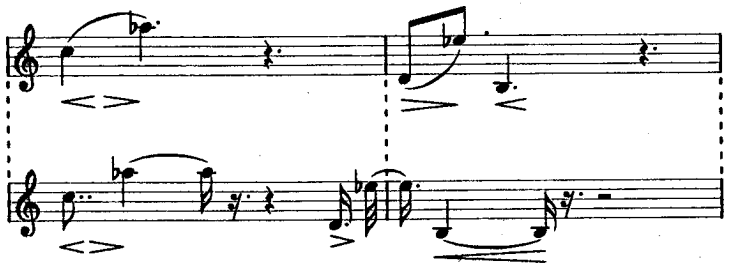



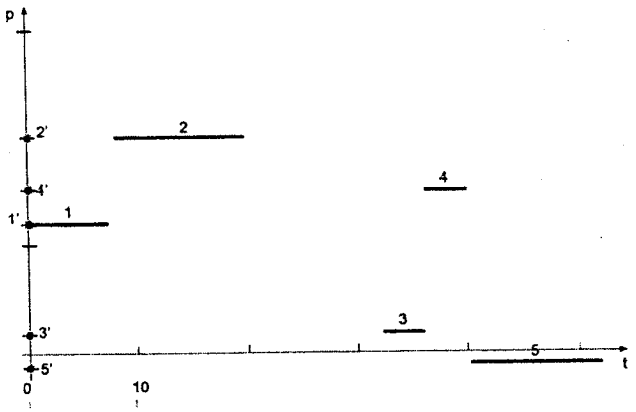
Fig. 3.123b

Through transformation, the durations are compressed, being given a duration d, $d' = d\cos\alpha$.

$$zFv = 100,\ \alpha = 90°,\ (x, y)P = 0, -1,\ \beta = 90°$$

All the sounds become sound points on the axis.

Fig. 3.124a



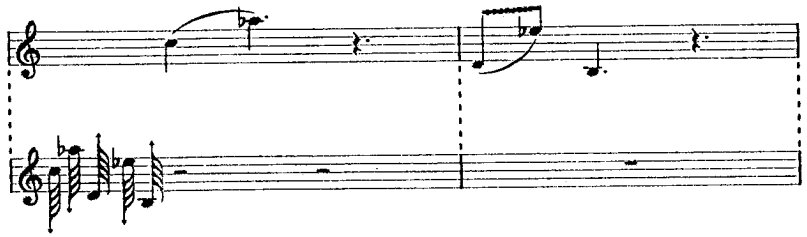
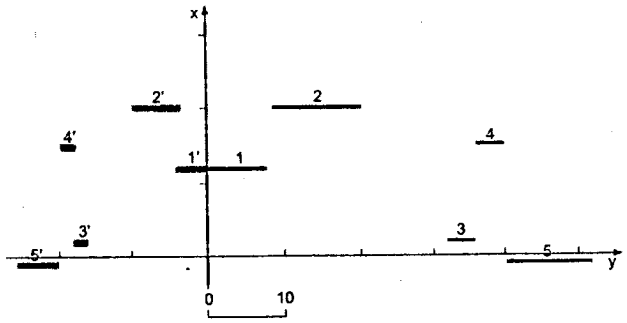


Fig. 3.124b

$zFv = 100, \alpha = 90°, (x,y)P = 0, -1, \beta = 120°$

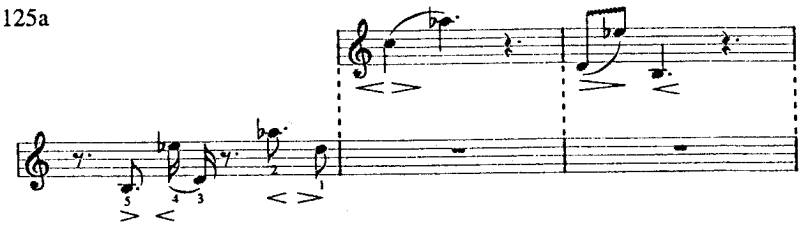
Fig. 3.125a



Inversion and 50% compression of durations (cos 120° = 0.5). When $\beta = 150°$, the image would give the symmetry of example 3.123a.

cos 150° = –cos 30°. Sound sense is changed, as well as the sense of dynamic signs.

Fig. 3.125a



Inversion and "compression" of durations.

$zFv = 100, \alpha = 90°, (x, y)P = 0, -1, \beta = 180°$

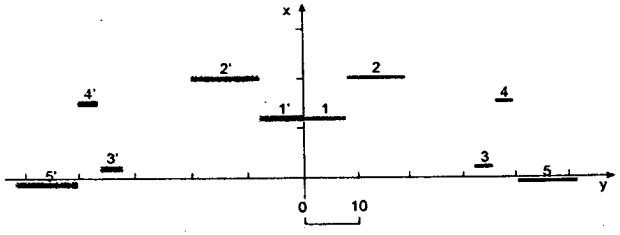
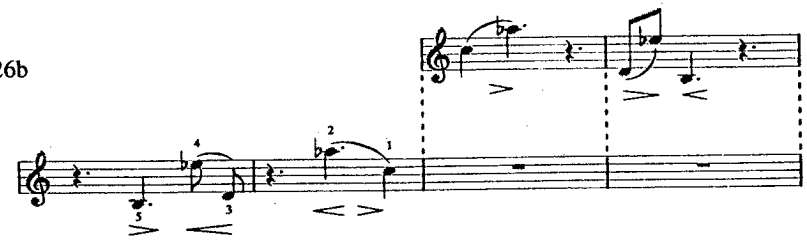


Fig. 3.126a

The classical "retrograde motion" (inversion): symmetric image as related to the axis perpendicular to the time axis.

Fig. 3.126b



Classical Inversion!

$zFv = 100, \alpha = 0°, (x, y)P = 0, -1, \beta = 180°$

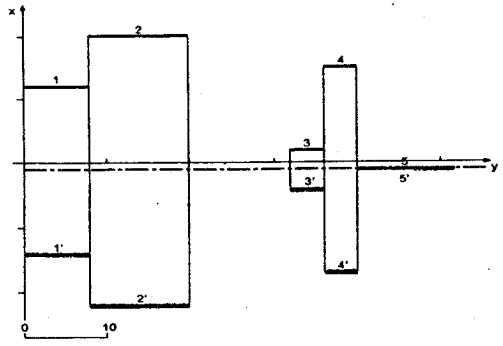


Fig. 3.127a

Reversion with respect to sound S_5 (B_2) (base of the sound form space (1.1.5))

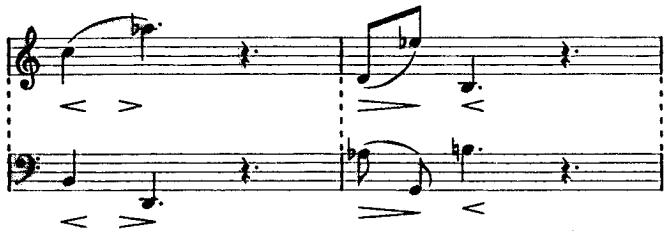



Fig. 3.127b

Reversion in terms of sound S_5

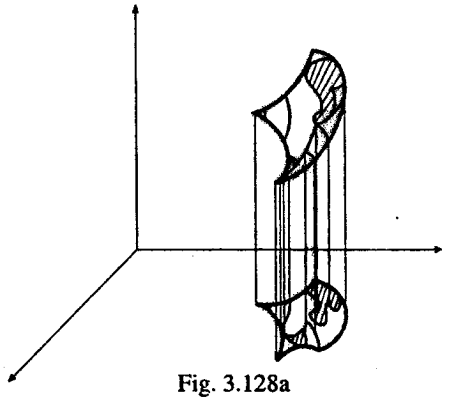


Fig. 3.128a

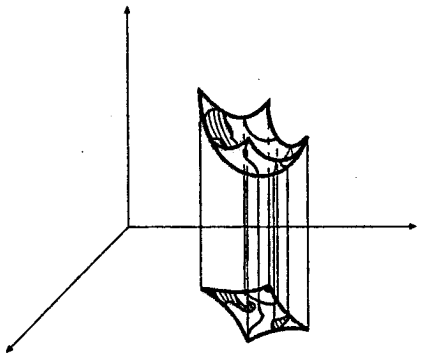


Fig. 3.128b

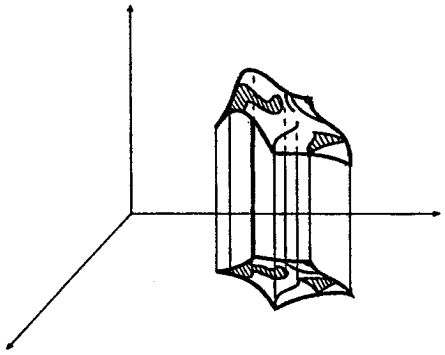


Fig. 3.128c

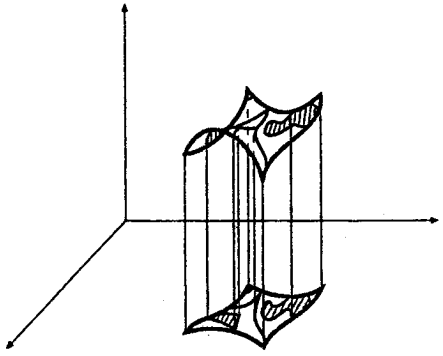


Fig.3.128d

4. CONTINUOUS TOPOLOGICAL TRANSFORMATIONS AND GENERATIVE MECHANISMS

4.1. DYNAMIC ASPECTS IN STUDYING AND GENERATING SOUND FORMS

4.1.1. General remarks

Throughout the previous chapters, sound forms have been examined from the static point of view of the relationships established between a finite number of sound events represented in an n-dimensional space, the dimensions of this space being the physical parameters characterizing a sound.

In this interpretation, the sound form is a part of an m-dimensional Cartesian product obtained from a set of parts of n-dimensional parallelepipeds, each such parallelepiped representing the Cartesian product of n finite intervals defined on the coordinate axis (1.1.3). But sound forms are *in time*-evolving phenomena; the part of the m-dimensional parallelepiped representing the sound form, is being "formed" *in time*., i.e., the Cartesian products characterizing the sound form depend on time t, which is one of their parameters.

The correspondence between sound sources action – ondulatory phenomena – and sound events perception implies the fact that any transformation *in time* of the values of sound sources physical parameters (frequency, amplitude, timbre) results in a transformation *in time* of the values of the parameters of the *sensation* of the respective sound events (pitch, intensity, timbre).

The value of the sound event parameters may vary upon a $[t_0, t_1]$ time interval, the sound form appearing as a transformation in terms of t, continuous on this interval.

Let A be a part of an m-dimensional parallelepiped representing a sound form SF unfolding during the time interval $[t_0, t_n]$.

By sectioning A at a t_x moment, $t_x \in [t_0, t_n]$ with an m–1 dimensional hyperplane perpendicular to the time axis, a part of this hyperplane will represent the Cartesian product of the points defining the values of the sound parameters of the sound form SF at the moment t_x. We will refer to

this part of the m–1 dimensional hyperplane (where time is the m-th parameter) as the sound hyperplane $H\,A_x$; SF will appear as arising at the t_0 moment and being "formed", "growing" in the sound space depending on time t. From this point of view, the sound form may be expressed as the sum of a set of the power of continuum of parts of sound hyperplanes perpendicular to the time axis unfolding in a finite time interval, each of these parts of hyperplanes representing the sound form at a certain moment.

4.1.2. Aristotle's vision of the correspondence between motion and form and its possible interpretation in the field of sound forms

The correspondence between the continuous transformation on a $[t_0, t_1]$ time interval of the values of the parameters of the oscillatory motions of some physical objects generating sound waves and the continuous transformation of the sound form grasped by the consciousness on the same time interval allows the confirmation in the acoustics field of Aristotle's postulate on the correspondence between motion transformation and form transformation, stated by him as:

"Movement's motion is form transformation into another form."
(Aristotle, *Physics*, Book V, Chap. 2.)

Indeed, when we know Aristotle's concept about the non-existence of motion out of time ("There is no time without motion," Aristotle, *Physics*, Book IV, Chap. 11;"... any motion exists in time," *ibid.* IV,14;"... for motion is measured by time and the time by motion," *ibid.* IV,14), and that, in his conception, motion means *transformation* ("Any motion is a certain modification," *ibid.* V, 1;"...there is no difference ... between motion and transformation," *ibid.* IV,11), the statement "movement motion is form transformation into another form" may be expressed by the phrase "motion's transformation implies form transformation." This sentence stating the correspondence between sound source parameter transformation and sound form transformation makes possible the confirmation in the field of sound forms of Aristotle's fundamental principle regarding visual forms.

4.1.3. Dynamic aspect of sound forms

The "formative," progressive character of the sound form and the fact that its "body" is obtained through a family of "sound images" continuously

generated within a given time interval, guide us to the possibility of defining sound forms under their *dynamic* aspect.

As stated (1.1.2; 4.1.1), from the point of view of a psychological phenomenon the sound form appears as the total sum of relationships the consciousness establishes between a finite number of sound events at a given moment; mathematically, these relationships may be expressed as a sound hyperplane in the (m–1) dimensional sound space. As a result, we may write:

A sound form appears as a family of the power of continuum of homeomorphic images obtained through a homeomorphism continuously acting upon a part of a sound hyperplane in a given time interval.

The previous definition presents the dynamic aspect of the sound form, aspect under which it will be henceforth considered.

4.1.4. Graphical representation of the dynamic aspect of sound forms. Sound form interpretation as generated by the track of a hyperplane in motion

The concept of curve in $\mathbf{R}^n$ is defined by means of an interval $[a, b]$ mapping in $\mathbf{R}^n$. It is worth recalling that, in a less rigorous but more suggestive way, the mathematician Camille Jordan represents a curve in a space $\mathbf{R}^2$ as being generated by the "track" of a point in motion. Of course, with this interpretation, the motion of the point is supposed to be continuous, hence without jumps, corresponding to a continuous mapping of the interval $[a, b]$ in $\mathbf{R}^2$.

We may imagine a straight line or a plane in motion in a space $\mathbf{R}^3$ describing a surface or a parallelepiped, respectively. More generally speaking, a part of an m–1 dimensional sound hyperplane may describe an m-dimensional parallelepiped in $\mathbf{R}^m$. This representation could grasp the "formative," the "progressional," aspect of the sound form in the m-dimensional sound space by the "track" left by a part of a sound hyperplane in motion in this sound space.

4.1.5. The formative aspect of the musical language

In order to examine music with respect to the formative aspect of sound phenomena, we will again reduce the sound space to a bidimensional space whose coordinate axes Ox, Oy represent the time and the pitch dimensions, respectively.

Resuming the representation given by Jordan to a curve in $\mathbf{R}^2$, a sound in the real sound space $\mathbf{R}^2$ can be viewed as being generated by the track of a *sound point M* in motion. We will refer to this curve as "sound trajectory" (*T*) (Fig. 4.1).

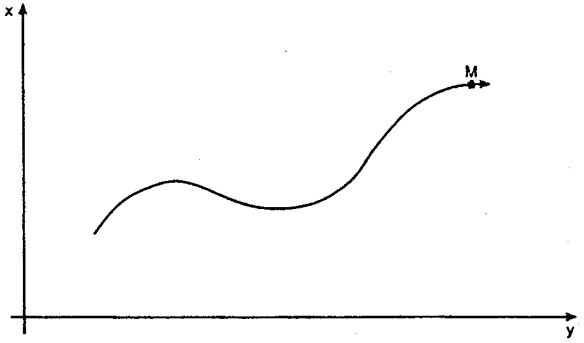


Fig. 4.1

4.1.6. Any sound event may be generated by the track of a sound point in motion in the real sound space

Let us consider a connected set of points not having the form of a segment of curve or of a straight line and representing a sound event in sound space $\mathbf{R}^2$ (Fig. 4.2).

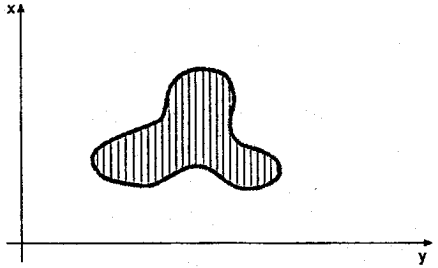


Fig. 4.2

This sound event may also be interpreted as being generated by the track of a sound point in motion taking into account the following:

1. Mathematical analysis knows the existence of curves filling a square or another surface, the so-called curves of Peano.

2. Discrete sound stimuli very close in time are perceived as continuous phenomena.

Therefore, two sound events represented in sound space, one described by a connected set of sound points and the other by a sound trajectory filling a surface whose outline is similar to that of the first set, are perceived identically.

As a result of the above, we may write the following

Remark

Any sound event represented in real sound space may be interpreted as being generated by a sound point in motion in this space on the condition that the trajectory it describes pass only once through a given point. The sound point motion represents its continuous transformation through translation in $\mathbf{R}^2$. Since the curve described by *M* may be parametrically represented as $y = f(t)$, where *y* is a continuous function on the interval $[a, d]$ on the time axis *Ox* (2.1.1), the position of point *M* in motion may be expressed as $y(M) = f(t)$. This interpretation proves in a graphical way the formative character in time of the sound event.

4.2. GENERATING MECHANISM

4.2.1. Parameter values expressed in terms of time

The dynamic aspect of sound events, i.e., their generation in time, has suggested the idea of examining the process of creating musical sound forms through topological transformations acting upon some finite sets of sounds, the value of the transformation parameters being expressed in terms of time *t*.

Definition

We will designate by Generating Mechanism of sound forms a couple consisting of a finite set of sounds and a number of topological transformations acting upon them in terms of time *t*. Within various time intervals, different transformations can act individually or simultaneously as compound mappings.

4.2.2. Elementary and complex music sound forms

By musical sound forms we mean sound forms of various sizes and modes of organisation having musical sounds as elements. By musical sound we mean a sound event produced by a human voice or a musical

instrument. According to (1.1.3; 2.11) the musical sounds are represented in the real sound space $\mathbf{R}^2$ as sound events consisting of connected sets of sound points arranged, as a rule, as straight line segments parallel to the time axis.

By *elementary musical form* we understand a sound form consisting of a restricted number of musical sounds and having a significant musical "personality". When the sounds are a sequence arranged in the direction of the time axis, we deal with a "motif," a "theme" or musical phrase. The general term of "melody" may be given as an example of this type of musical form. When a number of musical sounds of different pitches occur as "packages," all the considered sounds having simultaneous appearance and disappearance moments, we then deal with a "chord." Between these two characteristic cases – "melody" and "chord" – a number of intermediate forms (arpeggios, various forms of "accompaniments," etc.) may occur.

We call a *complex musical form* a sound form obtained out of one or several elementary musical forms by means of various methods of organisation, processing and development. Cannons, fugues, sonatas and symphonies are all examples of musical forms of this type.

4.2.3. Three modes of generating elementary music forms through a sound point moving in sound space $\mathbf{R}^2$

4.2.3a: *by translation of a sound point on a trajectory. (A "melody" or a musical theme may be "written" by a sound point in motion.)*

Since any sound event represented in sound space may be interpreted as being generated by a sound point in motion (4.1.6), a curve representing in $\mathbf{R}^2$ a melody or a theme may be viewed as being generated, i.e., "written" by a sound point in motion (translation) on a trajectory. In this context, a sound point M in motion can describe the curve Cv_1 representing a certain sound as well as the curves Cv_2 and Cv_3 representing the theme of the Romanian popular ballad *Mioritza* and the famous theme of the last movement of Beethoven's Ninth Symphony, respectively (Fig. 4.3).

It is easy to remark that Cv_2 and Cv_3 may be considered topological transformations of Cv_1, as well as that the three curves are topologically equivalent.

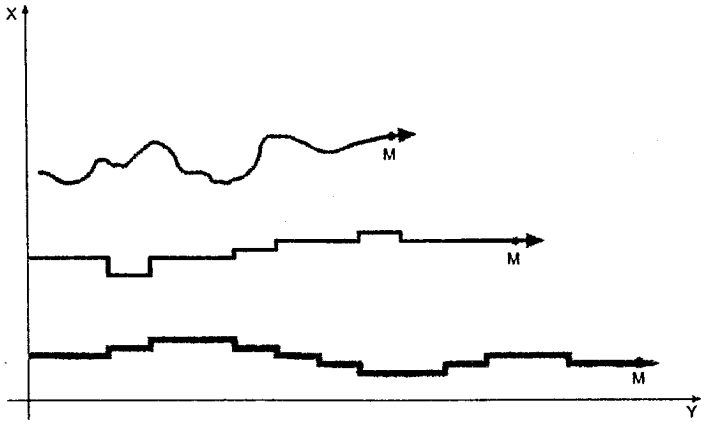


Fig. 4.3

4.2.3b: *by discontinuously surveying a sound trajectory image*

A sound trajectory can generate an infinity of musical themes consisting of sets of null and non-null sound points belonging to the trajectory.

We have already shown that a sound event becomes a null sound event if during a time interval $[t_x, t_{x+1}]$, the frequency of the wave phenomenon producing it has zero value.

The act of nullifying a sound event at a given moment or on a given time interval can be viewed as a transformation of the considered sound event. In a two-dimensional space, time and pitch being the space dimensions, a discontinuity of a curve representing a sound may be interpreted as a transformation of the sound into a null sound within the time interval defined by the discontinuity curve. Instead of defining the time interval wherein a sound trajectory describes a null sound, it appears more advantageous to define the time intervals when the trajectory describes the non-null sound, in other words, to define the complementary set of the subset of null sound points contained in a given sound trajectory (Fig. 4.4a).

Instead of defining the transformation of T on the $[t_{x1}, t_{x2}]$ interval into a null sound, we will define the non-null sounds described on the $[t_{x0}, t_{x1}]$ and the $[t_{x2}, t_{x3}]$ intervals, the sounds described by the sound trajectory *outside* these intervals being transformed into null sounds (Fig. 4.4b).

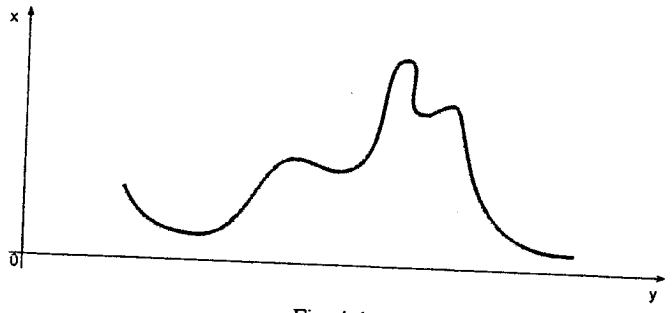


Fig. 4.4a

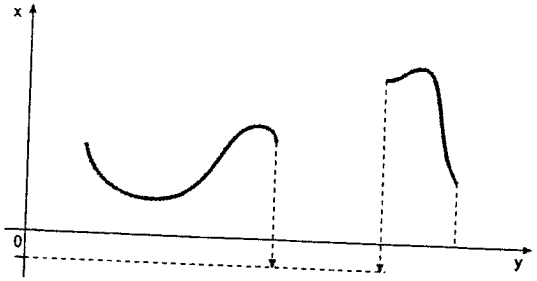
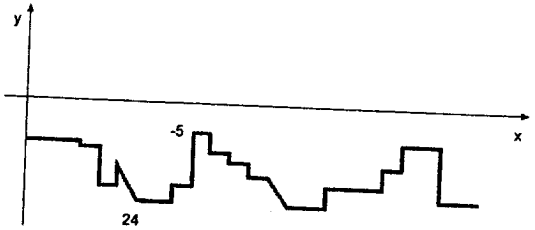


Fig. 4.4b

Let T be a sound trajectory described by a sound point in motion, i.e., a polygonal line passing through the points of coordinates x, y = (0, –7), (12, –7), (12, –8), (16, –8), (16, –16), (20, –16), (20, –12), (24, –19), (32, –19), (32, –16), (36, –16), (36, –5), (40, –5), (40, –9), (44, –9), (44, –11), (48, –11), (48, –16), (52, –16), (56, –20), (64, –20), (64, –16), (76, –16), (76, –12), (80, –12), (80, –7), (88, –7), (88, –19), (96, –19) (Fig. 4.5a).

Fig. 4.5a



By picking up the image of T at the moments x = 16, 26, 36, 50, 78, we will obtain T in the shape of the sound points (Fig. 4.5b) of coordinates x, y = (18, –16); (26, –14); (36, –5); (50, –14); (78, –12).

Here are, in musical notation, the five sounds of "null duration"[1](Fig. 4.5c).

[1] As mentioned in the composing process, the composer could give "duration" to sounds, in the sense of assigning them values of "sixteenth note" or "thirty-second note" and adequately diminishing the null or non null sound values "left" and "right".

Fig. 4.5b

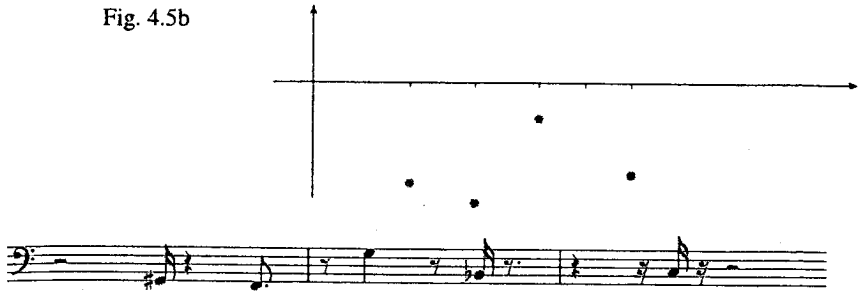


Fig. 4.5c

Preserving only the parts of the sound trajectory T defined by the intervals [14, 18], [26, 30], [46, 50], [58, 62], [78, 82], [88, 92], we get Fig. 4.6a

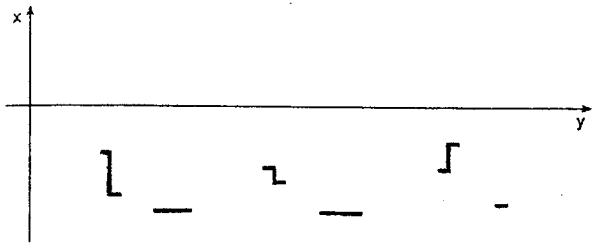


Fig. 4.6a

and in musical notation, the fragment in Fig. 4.6b.

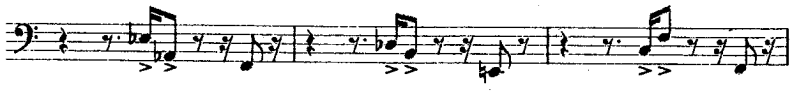



Fig. 4.6b

Going on in the same way for the time intervals [21, 23], [41, 43], [53, 55], [74, 76], we have Figs. 4.7a and 4.7b.

Fig. 4.7a

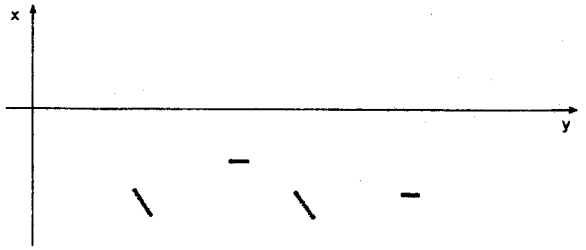


Fig. 4.7b

As the number of decisions in selecting the segments to be transformed in null sounds is actually infinite, the number of musical fragments that may be obtained from a given sound trajectory is also infinite. In some cases, in the set of musical fragments that can be generated through such topological transformations, certain trajectories may include segments which, in musical notation, reveal themselves to be fragments of musical masterpieces. For instance, by picking up on T the segments described during the time intervals [16, 20], [24, 36], [48. 52], [56, 68], we obtain the initial bars of the second part (Adagio) of Anton Bruckner's Symphony no.1 (Fig. 4.8).

Fig. 4.8

Allowing for the fact that "rests" represent null sounds, i.e., sounds of zero frequencies, the above-mentioned musical fragments are topologically equivalent, representing equivalent topological curves (1.2.2).

4.2.3c: *by the union of some points or segments belonging to a sound trajectory with their images obtained through various topological transformations*

To a point in the "journey" in sound space $\mathbf{R}^2$, various other transformations may also be associated.

Let P be a sound point describing the trajectory T in its motion in sound space $\mathbf{R}^2$. At any moment t_x of its motion (translation) on T we may associate to P another point P' through a certain transformation.

In Fig. 4.9 P is subjected to a projection from the projection centre C_p on the straight line D:

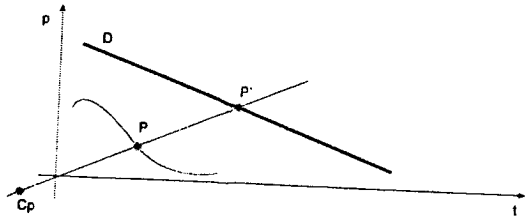
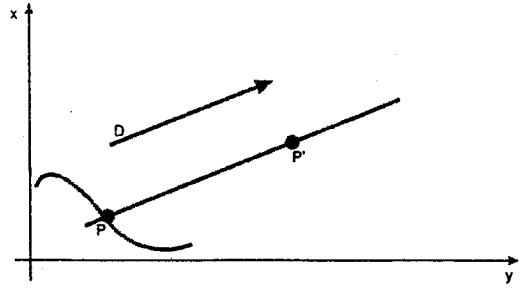


Fig. 4.9

In Fig. 4.10, to a translation along a vector parallel to D:

Fig. 4.10



In Fig. 4.11 to a homology with respect to an axis D, a centre C and a cross ratio k, $0 < k < 1$:

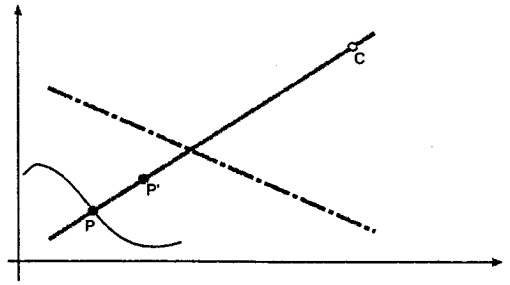
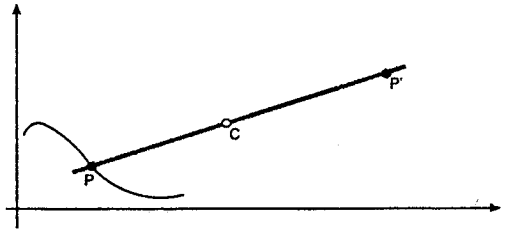


Fig. 4.11

In Fig. 4.12 to a homothety of center C and ratio q, $0 < q < 1$:

Fig. 4.12



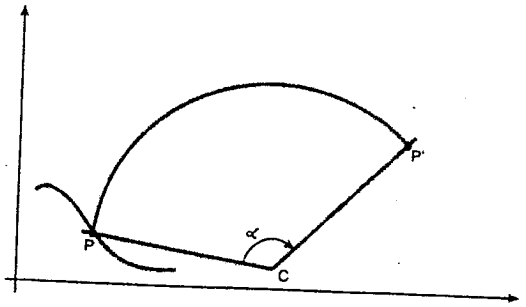
In Fig. 4.13 to a rotation around a center C with a rotation angle α, α=125°.

Let P be a sound point describing the sound trajectory T, defined by the function $y = 36 \sin \pi$, the unit on the Ox axis being equal to $\pi/18$.

Let us associate to P in its motion on T at the t_x moments its images through homotheties of centres C and homothety ratios k.

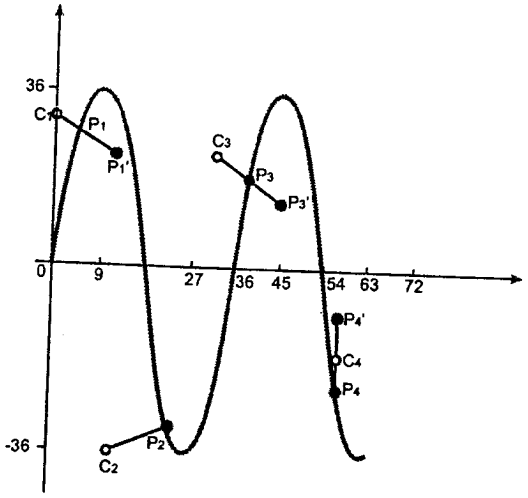
Let t_x = 5, 24, 39, 58; let the coordinates of the homothety centres (y, x) = (31, 0), (–36, 12), (23, 32), (–17, 58); and let 2.3, 1.5, 1.8 and –2.3 be the values of homothety ratios q.

Fig. 4.13



By carrying out the respective calculations, we obtain the points $P_{1...4}$ of coordinates $(y, x) = (27.5, 5), (-31, 24), (18, 39), (-23, 58)$ with their respective images through transformation in points $P'_{1...4}$ of coordinates (23, 11.5), –23, 30), (14, 45), (–9, 58) (Fig. 4.14).

Fig. 4.14



Apart from the points P, by transforming the sound described by T into null sound (i.e., by picking up from T only the points P) and making the union of points P with P', we obtain in musical notation Fig. 4.15.

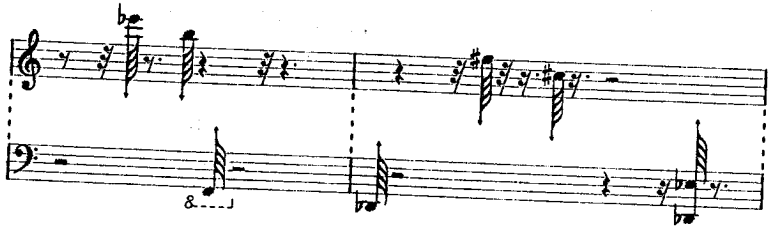



Fig. 4.15

4.2.4. The transformed segments of a sound trajectory produce a music fragment

Let T be a trajectory consisting of a polygonal line defined by the points of coordinates $(x, y) = (5, 8), (15, 10), (31, 22), (45, 15), (50, 7), (70, 0), (85, 19), (92, 26), (110, 17), (130, 30)$.

Let $I = [[i_1, i_2, i_3, i_4]]$ be a set of time intervals defined on the Ox axis, defined by the cuples of x values [20, 28], [54, 67], [85, 90], [115, 125], respectively, and let $\gamma = [[a, b, c, d]]$ be a set of transformations, namely:

(a) a homothety of centre of coordinates $x, y = (25, 0)$ and of homothety ratio q, $q = -2$;

(b) a rotation of an angle α, $\alpha = \pi/4$ (45°); around a center of rotation of coordinates $x, y = (80, 8)$;

(c) a projection from a projection center of coordinates $x, y = (74, 60)$ on a straight line D_1 defined by the point of coordinates $x, y = (117, 0)$ and angular coefficient m, $m = \text{tang } \pi/6$; and

(d) a translation parallel to a straight line D_2 defined by the point of coordinates $x, y = (142, 0)$ and the angular coefficient m_1, $m_1 = \text{tang } 7\pi/12$ (105°), the distance $AA' = DD' = 12$.

Let τ be a homeomorphism transforming into null sounds (zero frequency) the segments of T, excepting the intervals I.

By applying to each segment of T, corresponding to an interval from/a transformation from γ, i.e. $i_1 \rightarrow a$, $i_2 \rightarrow b$, etc., we obtain the following image (Fig. 4.16)

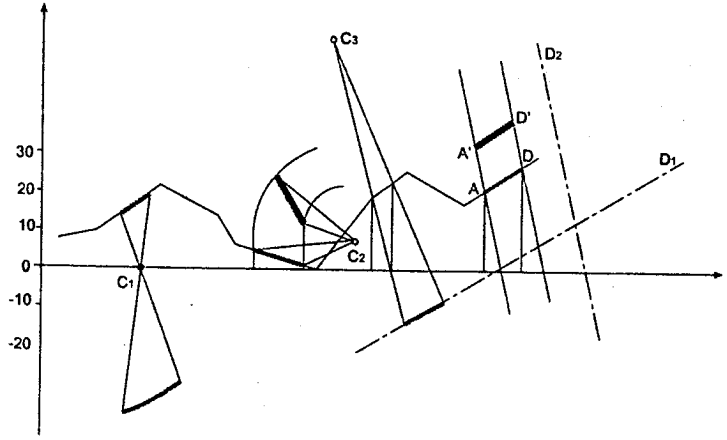


Fig. 4.16

Applying τ to the above image and carrying out the union of non-null segments of T defined by/with their images through τ, we obtain (Fig. 4.17).

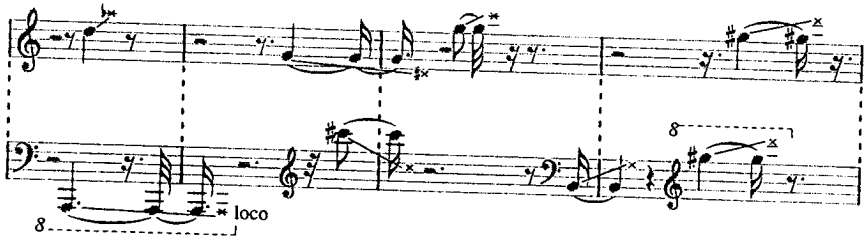



Fig. 4.17

4.2.5. A sound point through a compound transformation (translation compound with parallel homology) generates the accompaniment of a recitative belonging to a Bach cantata

A chain of chords obtained through an elaborated "ciphered bass" (1.5.5) may be interpreted as being generated by a trajectory in the shape of a polygonal line consisting of segments parallel to the coordinate axes Ox and Oy, described by a sound point M in motion in the sound space $\mathbf{R}^2$. This trajectory being transformed *in terms of time t* by families of parallel and orthogonal homologies with respect to a homology axis parallel to the time axis and a number of families K of parallel homology ratios, each *ratio* $k \in K$, $q > 1$, the difference between the numerator and denominator of each ratio expressing the values of an ascending musical interval, in semitones.

To illustrate the possibility of making up a "ciphered bass" by the above method, let us consider the fragment from the recitative "Wir ehren diese Herrlichkeit" from the cantata "Nun kommt des Helden Heiland" by J.S. Bach (Fig. 4.18)

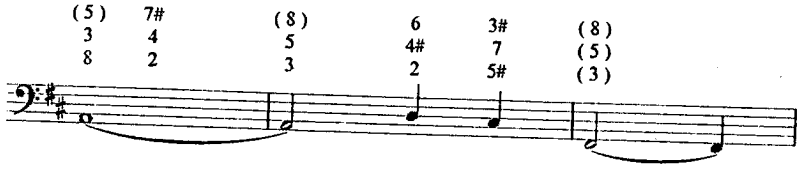



Fig. 4.18

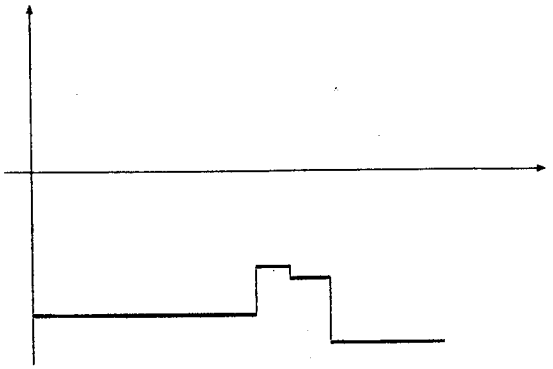
and the polygonal trajectory T (Fig. 4.19) defined by the points of coordinates $(x, y) = (0, -15)$, $(48, -15)$, $(48, -10)$, $(56, -10)$, $(56, -11$, $(64, -11)$, $(64, -18)$, $(88, -18)$, the segments parallel to the axis Ox corresponding to

the "bass" sounds and those parallel to the axis Oy representing null sounds (zero duration).

Let D be the homology axis defined by the point of coordinates x, $y = (0, -24)$ and angle α, $\alpha = 0$.

Let I be a number of time intervals corresponding to the sequence of musical durations of three times half-note, twice quarter-note and a dotted half-note, represented on the Ox axis by the [0, 16); [16, 32); [32, 48); [48, 56); [56, 64); [64, 88) intervals, (thirty-second note = 1) (Fig. 4.19).

Fig. 4.19



Let K be a number of families of parallel homology ratios equal to the number of intervals in I and selected so that, through transformation, a given point in T may be sent to a point in the sound space $\mathbf{R}^2$ with the same abscissa but having the ordinate equal to the musical interval indicated by the relevant "cipher" (Fig. 4.20). In order to express the intervals defined by families of figures and families of parallel and orthogonal homology ratios, we will go on as described under (2.4.5).

	(0, 16) I_a	(16, 32) I_b	(32, 48) I_c	(48, 56) I_d	(56, 64) I_e	(64, 88) I_f
	k_a	k_b	k_c	k_d	k_e	k_f
so-pra-no	$\frac{24+4}{24-15}$	$\frac{24+8}{24-15}$	$\frac{24+9}{24-15}$	$\frac{24+5}{24-10}$	$\frac{24+5}{24-11}$	$\frac{24+6}{24-18}$
al-to	$\frac{24+1}{24-15}$	$\frac{24+2}{24-15}$	$\frac{24+4}{24-15}$	$\frac{24-1}{24-10}$	$\frac{24-1}{24-11}$	$\frac{24+1}{24-18}$

tenor	$\frac{24-3}{24-15}$	$\frac{24-1}{24-15}$	$\frac{24+1}{24-15}$	$\frac{24-4}{24-10}$	$\frac{24-4}{24-11}$	$\frac{24-3}{24-18}$

By transforming each segment described by M on T corresponding to an interval in I by parallel and orthogonal homologies with regard to D and with the respective family of ratios from K, we obtain the image in Fig. 4.20.

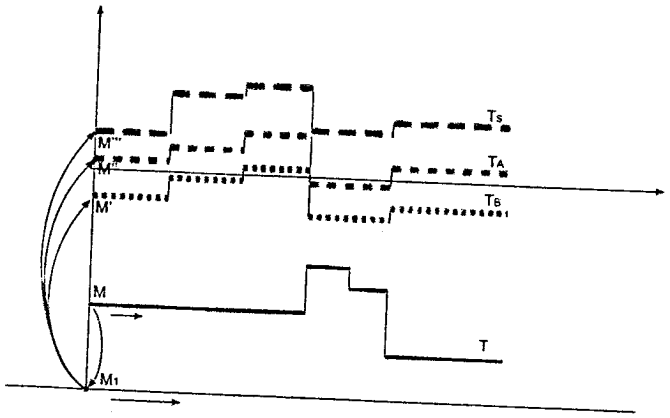


Fig. 4.20

The point M, through translation on T (bass) has generated through homologies the *sound matter* of *tenor*, *alto* and *soprano* voices.

Here is the fragment in musical notation (Fig. 4.21).

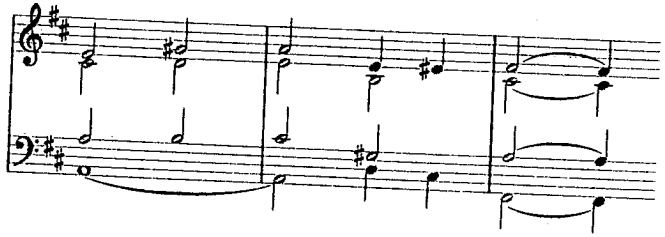



Fig. 4.21

In the composed transformations acting upon the point M when generating the above fragment, the value of each term's (translation, homology) parameters depends on time t

$y = f(t)$ (translation)

$k = f(t)$ (homology)

Here is, for example, the graph of the function defining the value q for obtaining the soprano voice (Fig. 4.22).

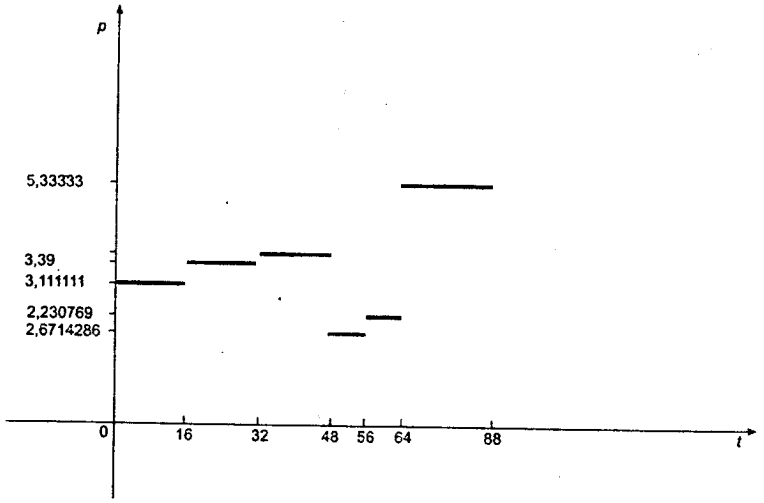


Fig. 4.22

4.2.6. A sound point in motion generates through the same composed transformations (translation, parallel homology) the *soprano* voice of Verdi's *Requiem* and the associated voices *alto*, *tenor* and *bass*

Let T be a polygonal trajectory defined by the points of coordinates $(x, y) = (0, 17), (32, 17), (32, 16), (64, 16), (64, 14), (96, 14), (96, 12), (112, 12), (112, 10), (128, 10), (128, 9), (144, 9), (144, 10), (160, 10), (160, 9), (176, 9)$ (Fig. 4.23a, b)

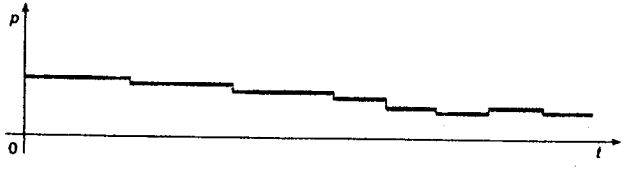


Fig. 4.23a

representing the soprano voice of a fragment from Verdi's *Requiem*.

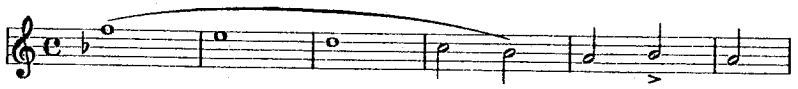



Fig. 4.23b

Let I be a sequence of time intervals, each interval corresponding to the duration of a segment parallel to Ox in T; let D be a homology axis defined by the point of coordinates $x = 0$, $y = 0$ and the angle α, $\alpha = 0$ and let K be a number of families of homology ratios $K = k_a, k_b, \ldots k_i$ corresponding to the number of segments parallel to Ox from T.

$$k_a = \begin{matrix} k_{a1} = 9/17 \\ k_{a2} = 0 \\ k_{a3} = 7/17 \end{matrix} \qquad k_b = \begin{matrix} k_{b1} = 12/16 \\ k_b = q_{b2} = 0 \\ k_{b3} = 7/15 \end{matrix}$$

$$k_c = \begin{matrix} k_{c1} = 10/14 \\ k_{c2} = 1/7\ (2/14) \\ k_{c3} = -1/2\ (-7/14) \end{matrix} \qquad k_d = \begin{matrix} k_{d1} = 7/12 \\ k_{d2} = 1/3\ (4/12) \\ k_{d3} = -7/12 \end{matrix}$$

$$k_e = \begin{matrix} k_{e1} = 7/10 \\ k_{e2} = 2/5\ (4/10) \\ k_{e3} = 7/10 \end{matrix} \qquad k_f = \begin{matrix} k_{f1} = 5/9 \\ k_{f2} = 5/9 \\ k_{f3} = -7/9 \end{matrix}$$

$$k_g = \begin{matrix} k_{g1} = 7/10 \\ k_{g2} = 1/5\ (2/10) \\ k_{g3} = -7/10 \end{matrix} \qquad k_h = \begin{matrix} k_{h1} = 5/9 \\ k_{h2} = 0 \\ k_{h3} = -7/4 \end{matrix}$$

The ratios are selected so that the homology sends points from T to points located at pitch intervals required by the traditional harmony laws.

By transforming T through parallel and orthogonal homologies with respect to D, each segment in T corresponding to an interval in I being transformed by a family of ratios from K, we obtain the image in Fig. 4.24a

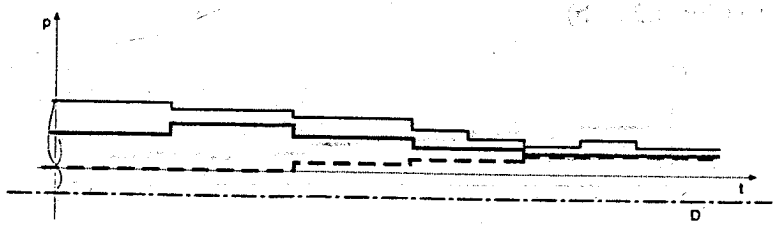


Fig. 4.24a

where T represents the *soprano* and its images by transformations represent the *alto, tenor* and *bass* voices, respectively.

In musical notation, the above-mentioned image corresponds to the "Pleni sunt coeli" fragment of Verdi's *Requiem* (Fig. 4.24b).

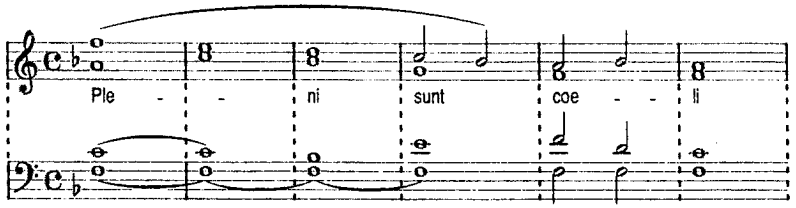


Fig. 4.24b

It is worth recalling that in the preceding examples the trajectory segments are *a component part of the generated musical form* as they represent the *bass* or the *soprano* of the considered musical fragment.

4.2.7. Musical fragments consisting exclusively of the images obtained through transformations applied to some points or segments belonging to a sound trajectory

Let us now imagine musical sound forms generated through the transformation of a trajectory or a part of it, but, unlike the above mentioned cases, the consequent sound forms do not contain trajectory points or segments but *exclusively* their images obtained through various transformations. We will refer to such a trajectory as virtual sound trajectory, or in brief, *virtual trajectory.*

In order to exemplify this case, let:

a) T be a virtual sound trajectory defined by the graph of the function $y = f(20 \cos x)$;

b) D_1 and D_2 be two straight lines defined by $y = 12$ and $y = 2.1$, respectively;

c) a straight line D_3 defined by the equation $0.47x - y + 10 = 0$ (i.e., defined by a point of coordinates $x = 0$, $y = 10$ and angular coefficient α, α = tang. 115° made with the Oy axis);.

d) C_{01} and C_{02} two projection centers located in the points of coordinates $x = \pi$, $y = 0$ and $x = 2\pi$, $y = 0$, respectively;

e) I_1, I_2 and I_3 three time intervals: $I_1 = (\pi, 5\pi/4)$, $I_2 = (5\pi/4, 11\pi/8)$ and $I_3 = (7\pi/4, 17\pi/8)$ (Fig. 4.25).

By projecting the segments of T corresponding to the intervals I_1, I_2 and I_3 from C_{01} on D_1 and D_2 and from C_{02} on D_3, we obtain the sounds S_1, S_2 and S_3 defined by the matrix:

1.29	–12	19.87	–12
4.31	–21	6.37	–21
5.58	12.60	6.56	13.06

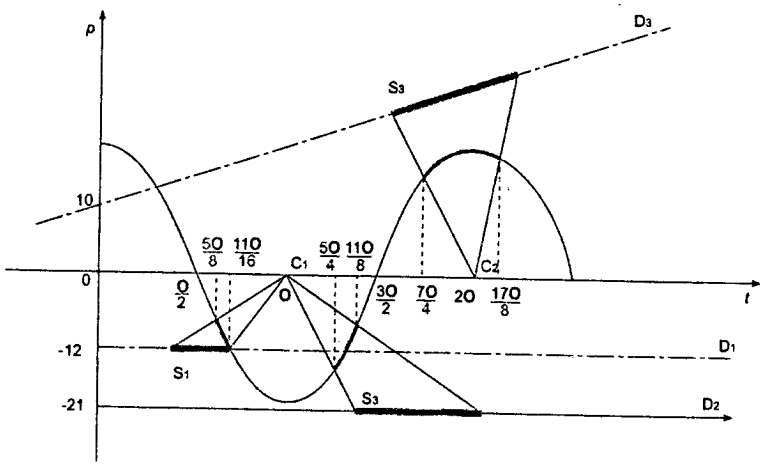


Fig. 4.25

The set of points defining the sounds S are the projections on D_1, D_2 and D_3 of the points x, y = (1.56, –7.65), (3.43, –14.14), (6.68, 18.48) belonging to T and in musical notation the sound sequence in Fig. 4.26.

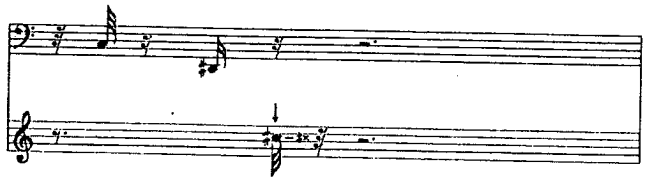



Fig. 4.26

Remarks:

1) The segments of the virtual trajectory T described by the function J projected on D_1, D_2 and D_3 within the time intervals I_1, I_2 and I_3 *are not a component part* of the musical form obtained through their transformation.

2) Unlike the case of generation of chords, the two transformations have been applied to two different segments of the trajectory and not simultaneously to one and the same segment.

Many examples may be given of musical forms having different complexities which may be interpreted as being generated through transformations applied, in terms of time, to certain segments of a virtual trajectory, forms which do not include the considered segments, but their images through transformation only. Musical "themes," "arpeggios," as well as certain forms of instrumental accompaniment may be interpreted as being generated in this way.

4.2.8. The concept of "function" in traditional music

Before approaching the study of generating the above mentioned musical forms, it is necessary to recall the concept of "function" in traditional music theory.

It is known that certain pieces may be classified according to the respective key in which they are created: C major, G minor, etc. The sounds of the respective scale are privileged as far as their distribution in the piece of music under consideration is concerned. Even more, in certain fragments of pieces having a specific traditional construction, for instance the sonata form, the almost exclusive distribution of these sounds becomes compulsory in the main themes or in the final bars.

From the examination of the methods of drawing up the *tonal* musical discourse, we note that, in certain sections, the composer uses as "basic material" only certain parts from the set of sounds defining the tonality of the considered section. Each of these subsets represents the sounds of a chord that may be built on the various "steps" of the respective scale. Traditional music theory shows that each sound (step) belonging to a key is provided with a certain "function": tonic (Ist step), dominant (step V), subdominant (step IV), etc. Any step may become the "fundamental sound" of a chord (basically built up with the third and fifth of the respective step); the set of sounds of the chord borrows the respective step "function". The chord may occur in "direct" condition or in different reversions without altering the chord sounds function.

The sounds may occur simultaneously as the chord proper or as a sequence displayed with respect to the time axis (e.g. a "melody," a "musical motif". The resulting whole section is considered to have the function of the fundamental sound on which the chord whose sounds form the respective sequence is built.[2]

[2] Sounds may occur in such sequences which actually do not belong to the considered chord, for instance "appoggiaturas," "delays," "passage notes," etc., without altering the sequence's function.

Arpeggios and certain types of musical accompaniment are included in this category of sound sequence. Music theory calls "arpeggios" the sounds of a chord successively evolving in time in increasing or decreasing order of the values of the respective sound pitches. When the chord sounds are displayed as ascending-descending pitch intervals, we deal with a "broken" arpeggio.

The following example shows arpeggios built with sounds of the dominant seventh chord belonging to the E flat major key (Fig. 4.27).

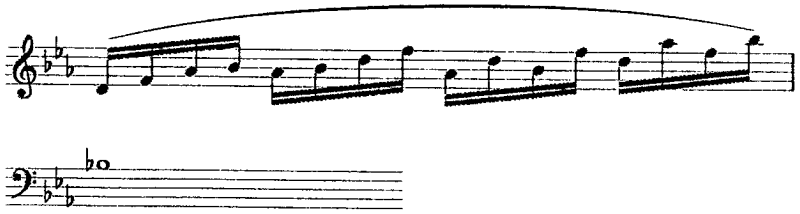



Fig. 4.27

The sequence may be extended through transposition to upper or lower octaves or by its iteration. With this last case, it usually serves as the accompaniment formula of a melodical line. Sometimes, it may even have thematic values, as, for instance, in Sonata Op. 27 No. 2 "Quasi una Fantasia" by L.v. Beethoven (Fig. 4.28).

Fig. 4.28

4.2.9. Transformation of a sound trajectory segment parallel to the time axis in an arpeggio with a certain tonal function

Let T be a straight line trajectory defined by the points $(x, y) = (0, -5)$, $(16, -5)$. Assuming that T represents the sound corresponding to the step V of the C major key, we may follow up in the process by which, through a family of parallel and orthogonal homologies related to an axis parallel to Ox, the dominant arpeggio of C major key may be obtained from T (Fig. 4.29a, b, c, d).

The family of homology ratios whose members were successively applied to a sequence of four time intervals defined on T ($I_1(0, 4)$, $I_2(4, 8)$, $I_3(8, 12)$, $I_4(12, 16)$) is identical with the family of ratios which, applied to T, would generate the major chord on the fundamental G sound (2.4.3).

Fig. 4.29a

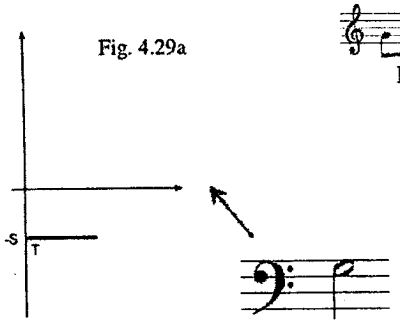


Fig. 4.29d

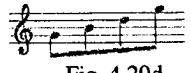


Fig. 4.29b

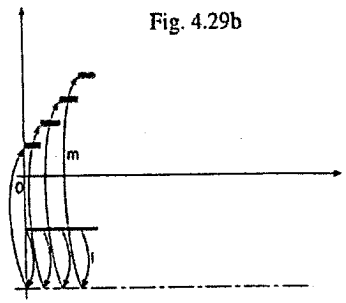


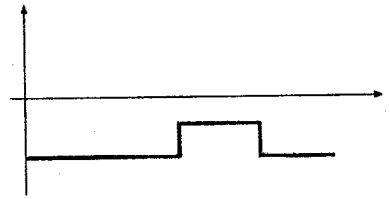
Fig. 4.29c

The above-mentioned possible interpretation of arpeggio generation, corroborated by the fact that an "arpeggio" or a finite set of successive sounds can "carry" a "tonal" function, allows the interpretation of certain fragments existing in instrumental music – "melodies" and their accompaniment – as being particular cases of musical sound forms obtained by transforming a virtual sound trajectory as indicated above.

4.2.10. Generation of musical themes and their accompaniment through topological transformations. Can Mozart's name be involved in such methods?

Let T be a virtual trajectory defined by the points of coordinates $(x, y) = (0, -12)$, $(32, -12)$, $(32, -5)$, $(48, -5)$, $(48, -12)$, $(64, -12)$ (Fig. 4.30).

Fig. 4.30



Let I and J be two interval sequences defined on the time axis Ox:

$I : I_1\ (0, -16);\ I_2\ (16, 24);\ I_3\ (24, 32);\ I_4\ (32, 48);\ I_5\ (48, 56)$
$J : J_1\ (0, 4);\ J_2\ (4, 8);\ J_3\ (8, 12),\ \ldots\ldots\ J_{16}\ (60, 64)$

Let K and M be two families of homology cross ratios:

$K : k_1 = 13;\ k_2 = 15;\ k_3 = 16.5;\ k_4 = 11.3$
$M : m_1 = 7;\ m_2 = 10.5;\ m_3 = 9;\ m_4 = 16.9;\ m_5 = 19.9$

Let us associate a homology cross ratio from K to each interval in I:

$i_1 \rightarrow k_1;\ i_2 \rightarrow k_2;\ i_3 \rightarrow k_3;\ i_4 \rightarrow k_4;\ i_5 \rightarrow k_5$

Let us associate a homology cross ratio from M to each interval in J:

$j_1, j_5, j_{13} \rightarrow m_1$; $j_2, j_4, j_6, j_8, j_{14}, j_{16} \rightarrow m_2$; $j_3, j_7, j_{15} \rightarrow m_3$;
$j_9 \rightarrow m_4$; $j_{10}, j_{12} \rightarrow m_5$; $j_{11} \rightarrow m_6$

We may say that the virtual sound trajectory T "treated" by homology on intervals I with the cross ratios from K will create a musical theme. "Treated" on the J intervals with the cross ratios from M it will create the theme's accompaniment.

4.2.11. Whereto leads the non-fulfilment of some conditions

We have shown in that, in order to obtain the considered transformation, homology should be parallel, orthogonal and the homology axis parallel to the time axis.

Let us suppose the three imposed conditions are disregarded.

Let $\alpha = -8^{\circ}$ be the angle of the homology axis with Ox defined in addition by the point of coordinates $x = 0$, $y = -14$, and the homology centre situated at a finite distance in the point of coordinates x, $y = (60, 60)$.

Here is the image in the sound space and its interpretation in musical notation (Figs. 4.31 and 4.32).

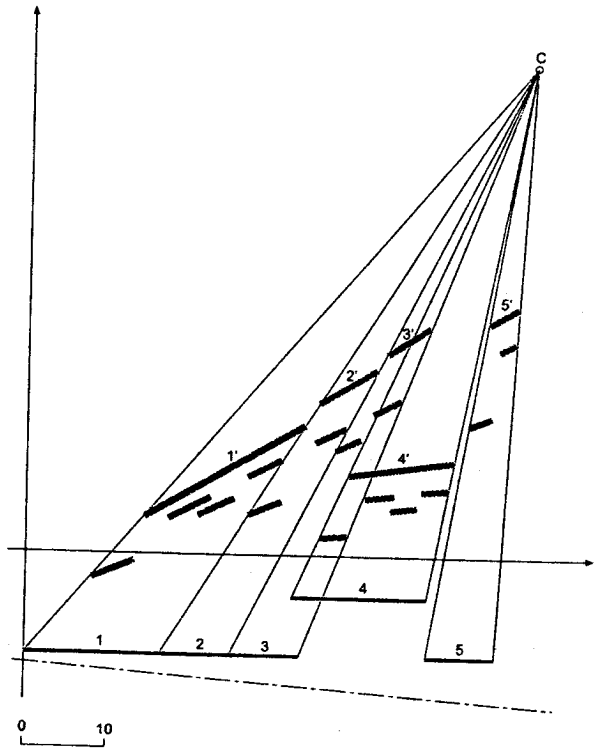


Fig. 4.31

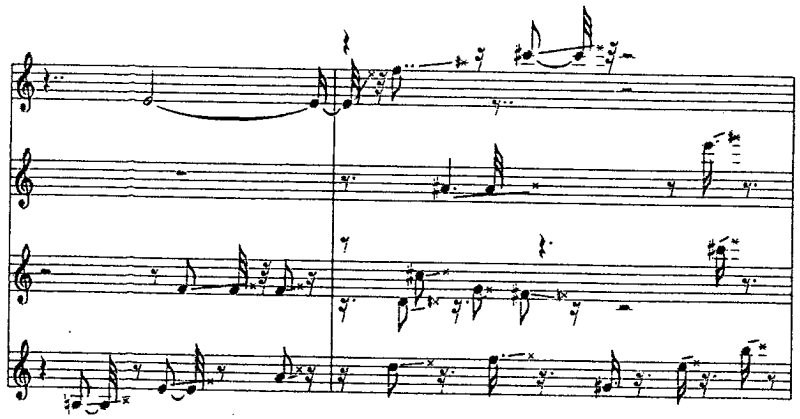



Fig. 4.32

Here is the same fragment subjected to the compound transformation *OD* * *AO*, i.e.. *Ordinate Lining up* (elimination of glissando type sounds through their transformation into constant pitch sounds) and *Abscissae Ordering* (cancelling overlapping) (3.1.3) (Fig. 4.33).

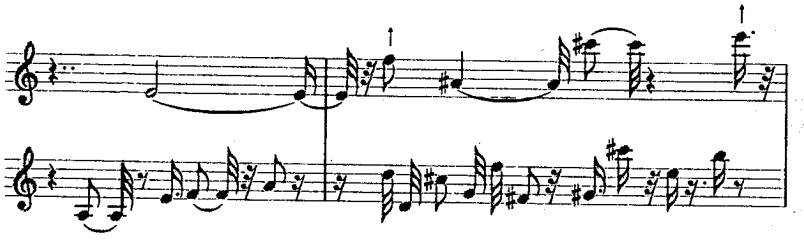



Fig. 4.33

Evidently, in a traditional sense, it is hard to agree that the images displayed in Figs. 4.35 and 4.36 represent a musical "theme" with its accompaniment.

Now we get a little closer to the required conditions. The homology centre is still at a finite distance (in the same point), but condition (3) is fulfilled: the homology axis is parallel to the time axis ($y = -14$). (Figs. 4.34 and 4.35).

Since by definition, if the homology axis is parallel to the time axis, homology does transform sounds of constant pitch into glissando-type sounds, the above image has been treated with *AO* (sound abscissae ordering) only (Fig. 4.36).

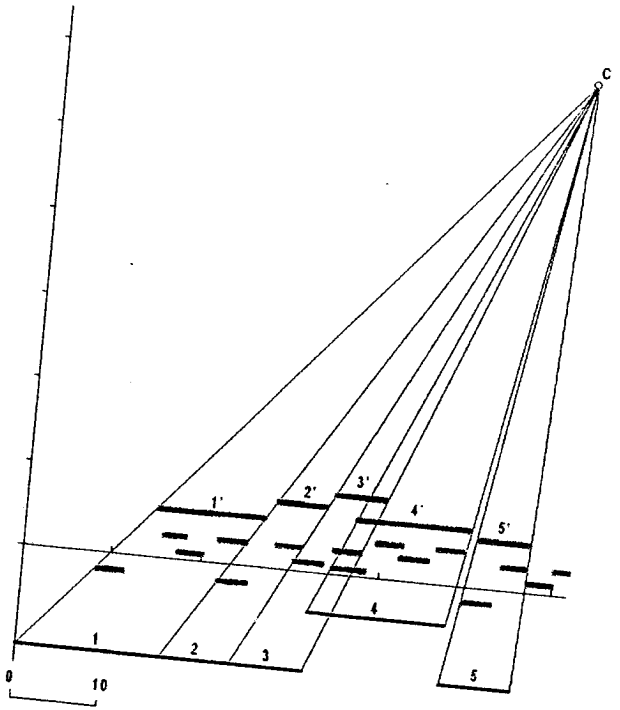


Fig. 4.34

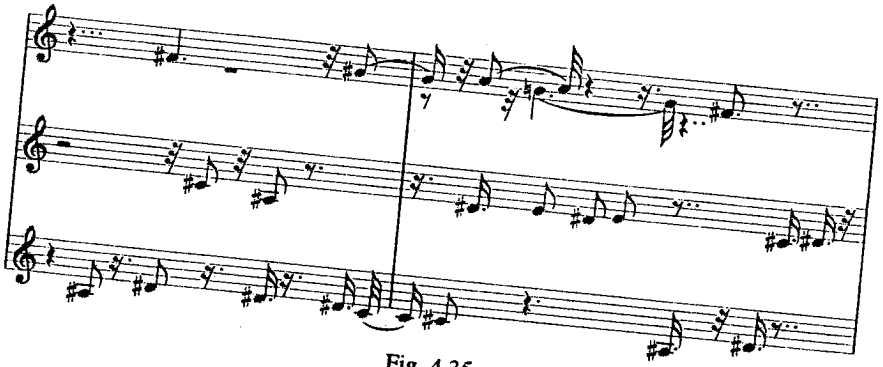



Fig. 4.35

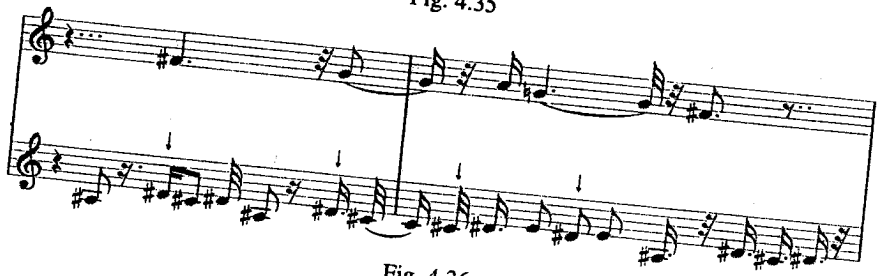
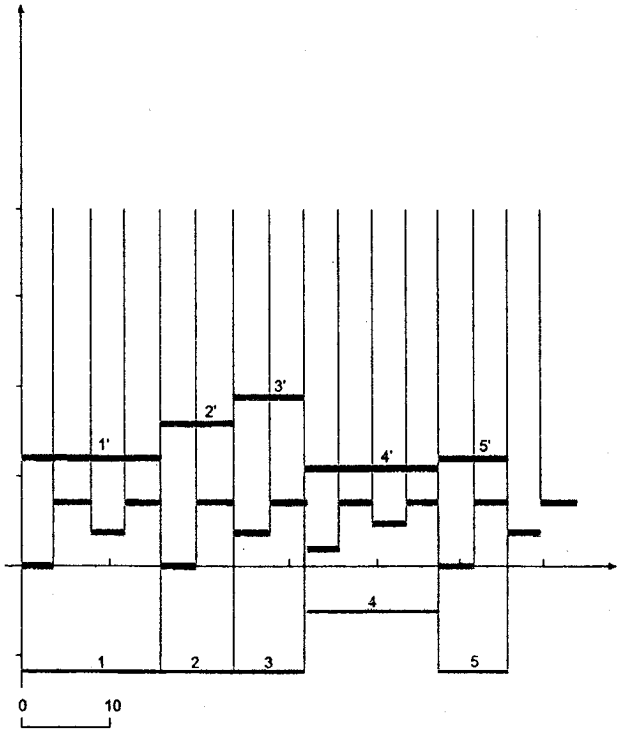



Fig. 4.36

4.2.12. A surprise for an enemy of mathematical thinking in music

Listening to the fragments shown in Figs. 4.33, 4.34, the second one could possibly be – indulgently – accepted even by an "enemy of inserting mathematics into the ineffable domain of music." By no means could he guess the surprise awaiting him when, in the preceding transformation, the homology centre "runs" to infinity and the projection lines become orthogonal (Fig. 4.37).

Fig. 4.37



The reading of the fragment in stave notation (Fig. 4.38) will reveal the first two bars of Mozart's well known C major Sonata.

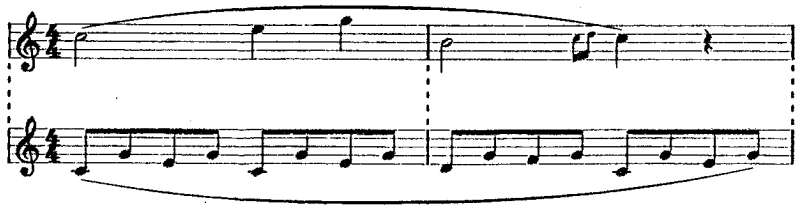


Fig. 4.38

4.3. MUSICAL STRUCTURES GENERATED BY THE DISCONTINUOUS SURVEY OF A SOUND FORM IN MOTION IN R^2

4.3.1. Sound forms in motion

Considering the sound point as the most elementary sound form (1.1.3) and admitting that sound is generated by the trace left by this form in motion in the sound space $\mathbf{R}^2$ (4.2.3), we will now extend this process by examining a sound form consisting of several musical sounds[3] in motion in this space.

It is obvious that the track of such a sound form in motion, in other words, the continuous "survey" of its position in sound space, will create a sound event of amorphousness and fluidity, devoid of any musical significance (Fig. 4.39).

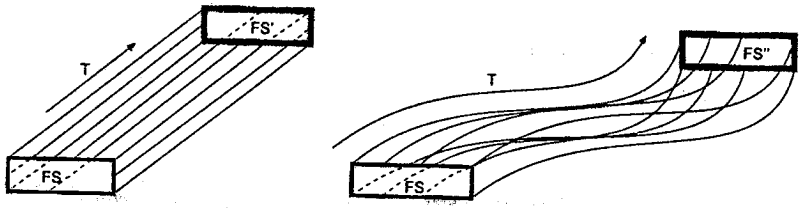


Fig. 4.39

In order to create a musical effect, the sound form in motion ought to be discontinuously picked up at various moments.

The sound form motion is examined with respect to a reference point, *RP*, arbitrarily chosen (center of figure, one of the rectangle's corners, etc.). In the following example, *RP* is the lower left corner of the rectangle defining the sound form.

Let T be the trajectory on which *RP* moves and let t_1, t_2, t_3, t_4, be the moments when the sound form is picked up (Fig. 4.40).

The movement of a sound form on a trajectory represents its transformation through translation.

Let $y = \dfrac{ax^2 - c}{b}$

[3] To have a clear image of this transformation, we will represent the sound form by its space as defined in 1.1.5.

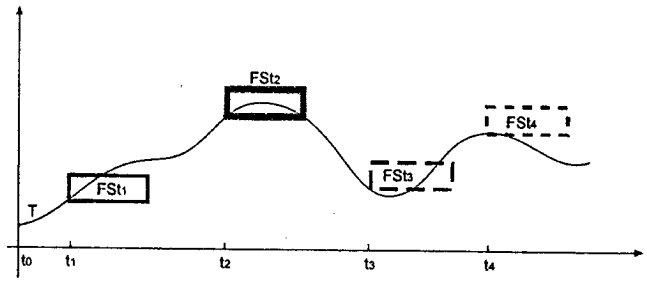


Fig. 4.40

where $x = t$ (time) and $y = p$ (pitch) is a function whose graph defines *RP*'s trajectory. The coordinates of *RP* being expressed in terms of time t, the position of the sound form in motion in $\mathbf{R}^2$ may be defined at any moment t_x.

4.3.2. Mathematical analysis in sound space. The musician, the derivatives. Rolle's, Fermat's and Lagrange's theorems

Faced with the algebraical symbol of a function whose graph represents a sound trajectory in sound space, the composer could ask himself the following questions:

1. Which are the pitch zones of the musical scale where the trajectory unfolds itself? Where and how much are they above or below the sound C_3?
2. On a certain time interval, does the sound trajectory have a "glissando" form or a constant pitch?
3. If it has a glissando form, is this ascending or descending? Does it have inflection points?[4]
4. Are the trajectory's maximum or minimum points situated in real or imaginary sounds zones, i.e., do they or do they not exceed the upper and lower audibility thresholds?

To find an answer to his questions, the composer should resort (even if he does not like it!) to mathematical analysis, i.e. to derivatives, to Rolle's, Fermat's and Lagrange's theorems. Considering the graphic representation of a function f as the track left by a sound point in motion in $\mathbf{R}^2$, Rolle's theorem shows the musician the following: if the function may be derivated on an interval I and the derivative is strictly positive, i.e., $f'(x) > 0$ $(x = t)$, then, on the interval considered, the sound trajectory is of the ascending glissando type;[5] but if the derivative is strictly negative, the sound trajectory

[4] We recall that a curve shows an inflection point when a tangent to the curve may be traced through that point and be parallel to the axis Ox.

[5] In this case we do not deal with rectilinear glissandos (2.1.2).

is of the descending glissando type. The examination of function f variation allows us to emphasise the number of real roots of the $f(x) = 0$ equation where the curve representing the sound "intersects" the sound C_3 (axis Ox).

Fermat's theorem states that at an extreme point of the function f, the derivative vanishes. Hence, for the composer, the moments of maximum or minimum sound pitch are in the set of roots of the derivative f' on the respective interval. Fermat's theorem also allows the discovery of whether or not a considered glissando type sound trajectory has inflection points.

Finally, the theorem of finite increases (Lagrange's theorem) shows that if the function f has a null derivative on an interval, then the sound trajectory described by the sound point in motion on the respective time interval has a constant pitch.

Since a sound trajectory is the "route" on which a sound form "travels" in sound space, the above concepts of mathematical analysis may be useful to a composer for discovering if, during a certain time interval, a certain sound form in motion on a trajectory "ascends" or "descends" on the pitch scale and for ascertaining the way this ascent or descent takes place if the maximum and minimum points of a trajectory do not somehow lead the sound form in motion outside the audible frequency range. In the end, by means of the same mathematical concepts one may notice whether the form motion is monotone and keeps the same pitch from the musical point of view. We recall these notions with the following example:

Let us consider the function $f(x) = x^3 - 6x^2 + 9x - 1$.

We notice that the field of definition of this function is the $(-\infty, +\infty)$ interval and the limit of function f when $x \to +\infty$ is $+\infty$ and when $x \to -\infty$ is $-\infty$.

The graph cuts the Oy axis in the coordinate point (0, 1). We do not solve the equation $f(x) = 0$, leaving on purpose the function graph to show us how many roots are real and where they are situated.

The function derivative is $f'(x) = 3(x^2 - 4x + 3)$.

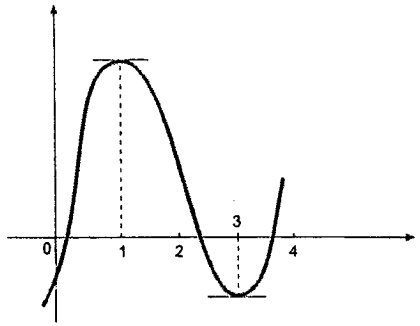
It has the roots 1 and 3. By means of these indications we may build up the function variation table representing the trajectory of a sound point in motion in $\mathbf{R}^2$ (Fig. 4.41).

x	$-\infty$		0		1		3		$-\infty$
$f'(x)$	$+++$				$++0---$				$0+++$
$f(x)$	$-\infty$	↗	-1	↗	3	↘	-1	↗	$+\infty$

Fig. 4.41

When x increases from $-\infty$ to $+\infty$ the function increases from $-\infty$ to -1 (for $x = 0$), further increases up to 3 (for $x = 1$), then decreases to -1 (for $x = 3$), then increases to $+\infty$. In the point $x = 1$ the function has a maximum equal to +3, and in the point $x = 3$ it has a minimum equal to -1; the function graph is plotted in Fig. 4.42.

Fig. 4.42



The graph gives indications about the roots of the equation $f(x) = 0$. As a matter of fact, it cuts the axis Ox in three points, x_1, x_2, x_3. Hence, the equation $f(x) = 0$ has three real roots x_1, x_2, x_3. A simple calculation shows that

$$f(0) = -1; f(1) = 3; f(2) = 1; f(3) = -1; f(u) = 3$$

Therefore, $0 < x_1 < 1; 2 < x_2 < 3; 3 < x_3 < 4$.

4.3.3. Musical significance of the variation of function $(f(x) = 3(x^2 - 4x + 3))$

Allowing for the unit of measure (u) on the time axis Ox, u = whole note (a bar of 4 quarter notes) and for the unit of measure (w) on the pitch axis Oy, $w = 12 \times 1/2$ tone (an octave), and considering the graph of the function as representing a sound described by a sound point in motion, from the musical point of view, the analysis carried out has the following meanings:

The sound appears from the zone of imaginary sounds of low frequency (F), $F < 16$ Hz and increases its pitch up to C_2 situated at the beginning of the first bar. The pitch further increases during the first bar up to C_6, then decreases during the next two bars until reaching C_2 at the beginning of the fourth bar when it increases towards the zones of imaginary sounds, $F >$ approx. 22.000 Hz. The ambitus (1.1.6) wherein the sound displays is of 4 octaves, and has as minimum and maximum points the sounds C_2 and C_6, respectively. The reference point of a sound form in motion on the trajectory meets the central C pitch (C_3) in the first, third and fourth bars (Fig. 4.43).

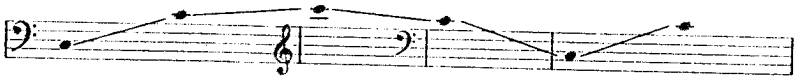


Fig. 4.43

The comparison of Figs. 4.42 and 4.43 emphasizes how imperfect the musical notation is as regards the "writing" of a glissando sound (2.5.1b).

4.3.4. The speed at a given moment of the unfolding of a glissando sound. The derivative assisting again the composer

The examination of the graph of the function shows that, at equal time intervals, the curve sections have different lengths, i.e., the sound point (M) moves on trajectory at different rates: for traversing the interval of four octaves $C_2 - C_6$ when "climbing", 4 quarter notes are needed, while when "descending" 8 quarter notes are required. Considering the metronomic value is quarter note = 60, for carrying out the "ascent" 4 seconds are necessary and for going down the same interval 8 seconds are used. It is easily apparent that between the sound point in motion and the pianinst's fingers running in glissando over the piano keys there is an isomorphism. But, how could we know the speed (v) of the sound point in motion, i.e., of the fingers on the piano at a given moment t situated between the glissando "ends"?

Of course, the answer is given by the value of the derivative of the considered function wherein the variable is expressed as the parametric form of t

$$v = 3(t^2 - 4t + 3).$$

What is the speed of the glissando display in the 2nd bar at the third "beat" moment?

As we know, the musical "bar" in Fig. 4.43 has 4 quarter notes which represent the unit of measure (u) on the Ox axis and the moment t has the value $t = 1.5$. Consequently, the speed (v) of the sound point in motion at the moment $t = 1.5$ is $v = -2.25$. The derivative is negative. At the moment $t = 1.5$ the sound goes down at a rate of 2.25 w (units on axis Oy) per time unit u. Since w is equal to 12 semitones, the sound point M in motion descends by 27 semitones in a bar of 4 quarter notes, i.e., by 6.75 semitons (roughly a fifth) per "beat." The metronomical value of a "beat" easily makes the connection between the sound point's speed rate and chronological time.

Let quarter note = 124; M will descend by roughly a fifth in 60/124 of a minute, i.e., in 0.483 ... second.The rule of three tells us that in this case M shifts by 7.233 semitones per second.

The speed of a sound form on a trajectory is useful for defining some generative mechanisms we will further take into consideration (4.5.1.).

4.3.5. A sound form "walking" on a curve in space R^2 generates imitations and canons

Let us return to section 2.4.1 to the particular cases of topological transformations used in music and to transformations through translation. It has been shown that a series of composition methods, i.e., imitation, repetition, canon and transposition may be interpreted as being transformations through translation related to a vector defined in the sound space $\mathbf{R}^2$. By generalizing we may say that these methods of composition can be interpreted as translations in relation to any curve in this space.

Let SF be a sound form in motion on a trajectory T and SF_i ($i = 1, 2, ..., n$) the SF images surveyed (picked up) at different moments t_x belonging to a certain time interval.

Unlike the cases examined under (2.4.1), where the transformation by translation of a sound form SF was considered from a given point of the translation vector to another given point, in the present case *the reference point of the form is "observed" during its continuous motion on a trajectory* T (motion defined in terms of time t). SF 's images may be picked up at any tx moment belonging to a time interval I on which the motion is observed. The trajectory set of points being of the power of continuum, the set of images that may be generated as a function of time (t) by the [T, SF] couple is of the power of continuum. The [T, SF] couple forms a *Generative Mechanism*. Depending on the form of T and on the moments of picking up SF's image, the generative mechanism "produces" imitations, canons, repetitions or transpositions. Which one of these forms is generated, depends on the relationships between the projections of the sound form FS on the time axis.

Let GM be such a mechanism: the [T, SF] couple is represented by the graph of the function

$$y = x^3 - 6x^2 + 9x - 1$$

and by the sound form shown in Fig. 4.44a and 4.44b.

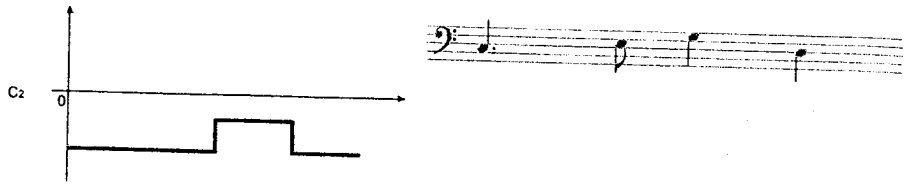


Fig. 4.44a

Fig. 4.44b

The motion of *SF* on *T* is illustrated in Fig. 4.45.[6]

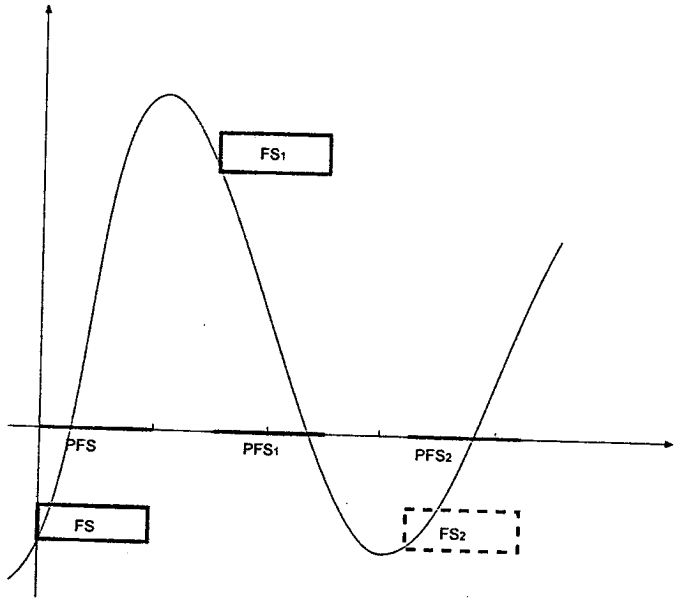


Fig. 4.45

By picking up the image of *SF* moving on *T* at moments $t = 0$, $t = 1.5$, $t = 3.4$*[7], (*SF*, SF_1, SF_2), the relationships between their projections (*PSF*) on the time axis have the form:

[6] The unit of measure on Ox is the whole note; on Oy is the octave. For greater clearness, the ambit of the sound form on the graph is reduced by 1/3. Also, since the zone wherein the trajectory moves belongs to sounds of low pitch, it was considered more avantageous to let axis *Ox* correspond to sound C_2.

[7] We recall that the unit of measure u (4..3.3) contains 32 thirty-second notes (a whole note). Hence, in Fig. 4.46 the decimal notation $t = 1.5$ must be interpreted as $t = u+1/2u$ (a whole note plus a half-note i.e., a dotted whole note); $t = 1.25$ in Fig. 4.49 must be interpreted as $u + 1/4u$ (a whole note plus a quarter note). Sometimes the decimal figure notating a certain point on the time axis does not correspond to an integer number of thirty-second notes. In such a case a round off is necessary; $t = 0.3$, $t = 2.2$ and $t = 3.5$ in Fig. 4.49 can be taken as examples.

$PSF \cap PSF1 = \varnothing$; $PSF1 \cap PSF2 = \varnothing$

The intersection of the considered projections are empty: *GM* produces imitations (Fig. 4.46).

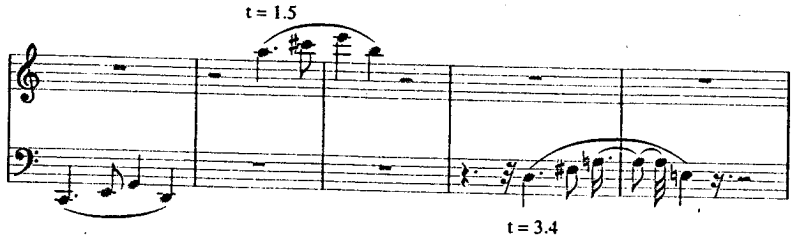


Fig. 4.46

Now, let the musical sound form *SF* (Fig. 4.47) be

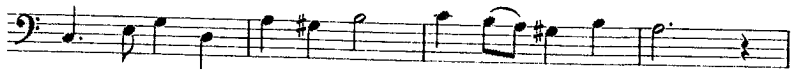


Fig. 4.47

moving on the trajectory represented by the graph of the same function $f(x) = x^3 - 6x^2 + 9x - 1$. By picking up the images of *SF* on *T* at the $t = 0.3$, 1.25, 2.2 and 3.5 moments, it is easy to notice that the intersections of their projections on Ox are non-empty:

$SFP \cap SFP1 \neq \varnothing$; $SFP1 \cap SFP2 \neq \varnothing$; $SFP2 \cap SFP3 \neq \varnothing$ (2.4.1α)

GM generates the canon on 4 voices in Fig. 4.48.

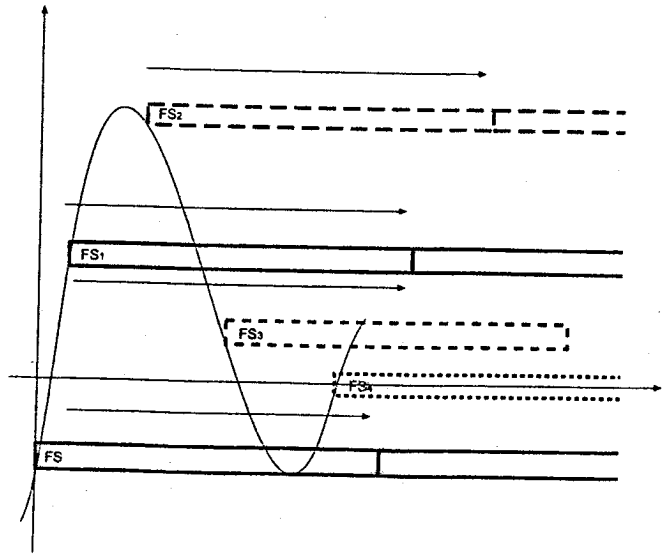


Fig. 4.48

Here is the image of this canon in musical notation (Fig. 4.49).

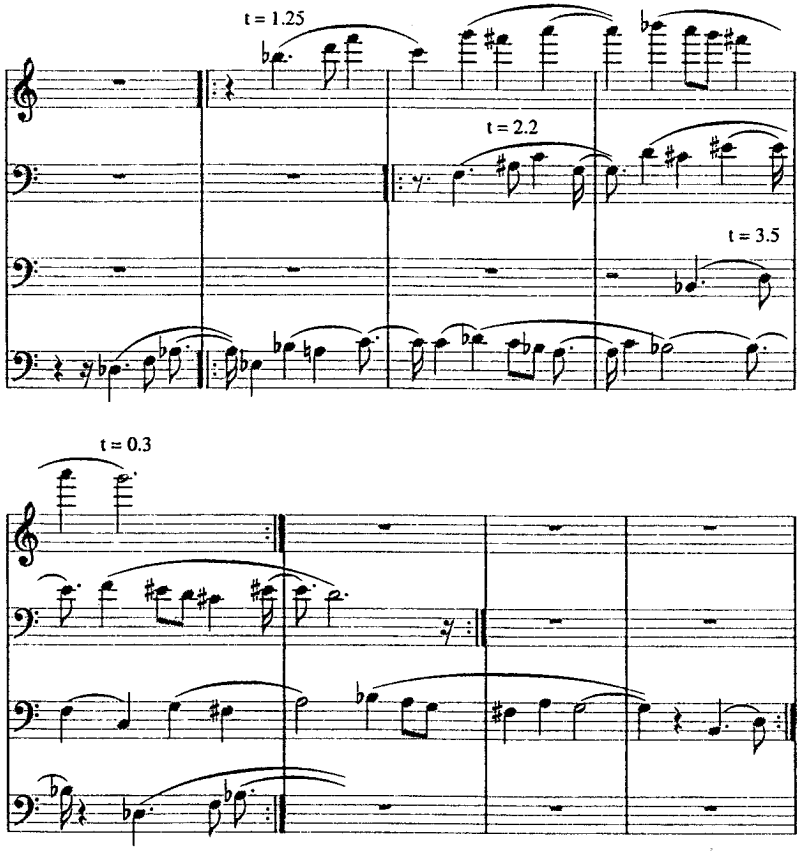



Fig. 4.49

GM cannot generate repetitions and transpositions since *T* does not contain segments which are parallel to the *Ox* and *Oy* axes.

In order to obtain a "continuous" canon, i.e., one which repeats itself a number of times, the SF_i images have to be transformed through translation with respect to a vector parallel to the *Ox axis* (Fig. 4.49). The *SF* duration being of 4 seconds (quarter note = 60), the repetition starts for *SF* at $x = 4$; for SF_1 at $x = 4.3$; for SF_2 at $x = 5.5$ and for SF_3 at $x = 6.5$.

The "repetition" signs have to be placed as shown in Fig. 4.49. Since the repetition of a musical fragment can be interpreted as a *transformation through translations* with respect to a vector parallel to the Ox time axis, the result is that a "continuous canon" can be viewed as a transformation of a sound form of the type shown in Fig. 4.48, compounded with a translation parallel to the time axis.

4.4. THE CONCEPT OF "FUNCTION OF TIME" AS REFLECTED IN THE THINKING OF PRECLASSICAL COMPOSERS. SIXTEENTH CENTURY MUSIC MAKES USE OF SPACE COORDINATES

4.4.1. Open and closed canons

Building canons by picking up at certain moments a sound form in motion on a trajectory, may seem strange at first. However, a certain mode of canon notation used sometimes by preclassical composers, shows that the idea these great masters of counterpoint had about "canon" was similar, almost identical to, the above-mentioned analytic mathematical one.

In preclassic music the canon is recorded in two ways:

(a) *open*, i.e., each voice is written down on the staff;

(b) *closed*, i.e., only the *antecedent*[8] is written, and during its unfolding, the "entrances" of the *consequent* (or those of the *consequents* when the canon is of several voices) are marked on the adequate moments by specific signs.

Here is a three-voice canon by Bernabei[9] in open notation (Fig. 4.50a).

The same canon in closed notation (Fig. 4.50b) trinus in una; diapente + diapason ++

The indication "trinus in una" means that the three voices have a common notation. "Diapente" and "diapason" are the old denominations of the fifth and octave pitch intervals. The signs + and ++ show the moments when the second and third voices "enter" at the specified intervals of fifth and octave.

The "closed" notation proves that preclassic composers had the intuition of a sound form "appearing" at certain moments and places on the staff (corresponding to the sound space $\mathbf{R}^2$) appearence indicated by the occurrence moments (+, ++ corresponding to abscissa ($x = t$)) and by an indication expressing a pitch – the interval diapente, diapason – corresponding to ordinate *y*, $y = p$.

[8] *Antecedent* and *consequent* in polyphonic language mean the initial theme and its imitation, at different time and pitch intervals.

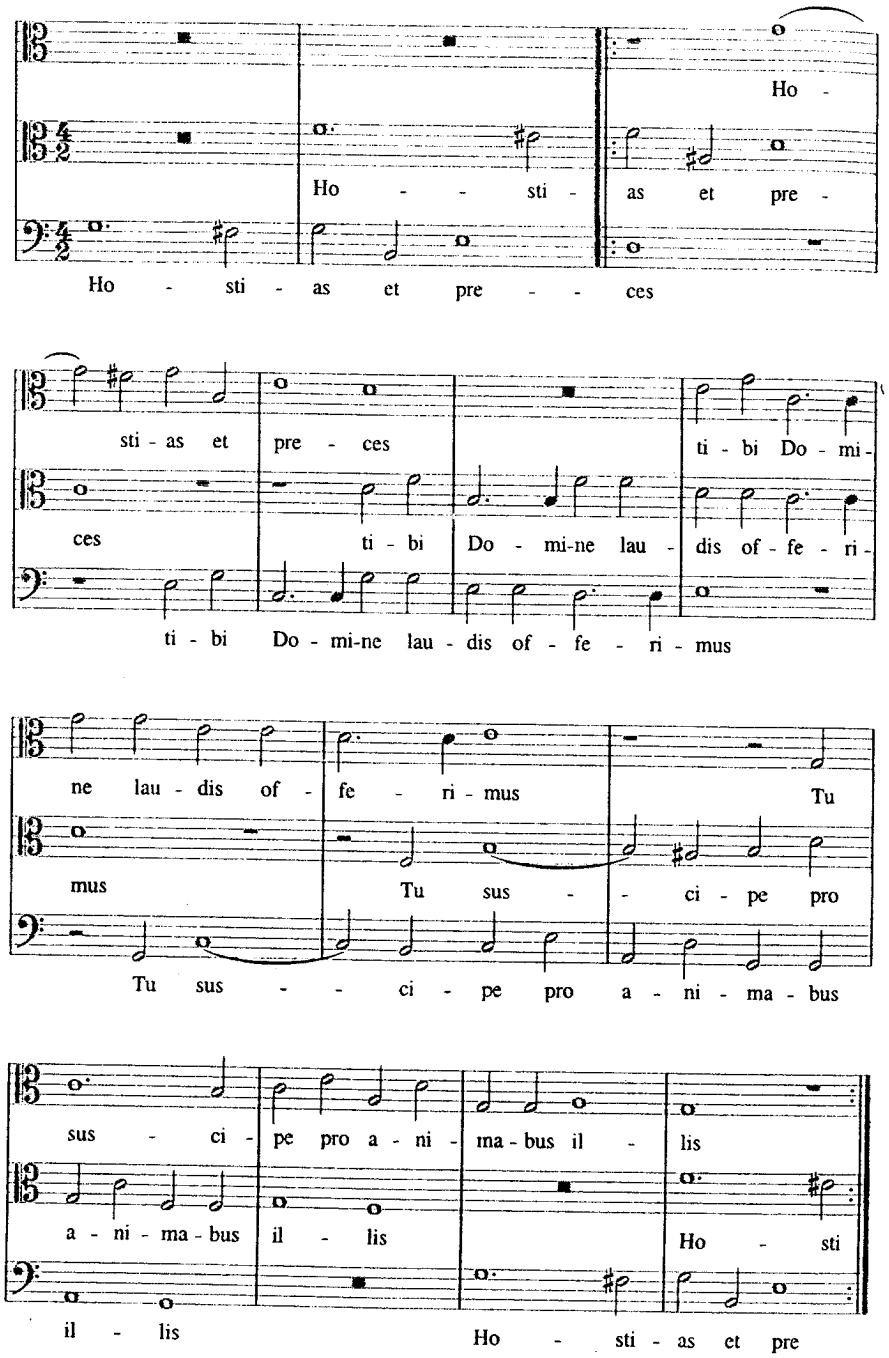
[9] Giuseppe Eriele Bernabei, Italian composer and organist (1620–1687).

Ho - sti - as et pre - - ces
Ho - - sti - as et pre -
Ho -
sti - as et pre - ces
ti - bi Do - mi -
ces ti - bi Do - mi-ne lau - dis of - fe - ri -
ti - bi Do - mi-ne lau - dis of - fe - ri - mus
ne lau - dis of - fe - ri - mus Tu
mus Tu sus - - ci - pe pro
Tu sus - - ci - pe pro a - ni - ma - bus
sus - ci - pe pro a - ni - ma - bus il - lis
a - ni - ma - bus il - lis Ho - sti
il - lis Ho - sti - as et pre

Fig. 4.50a

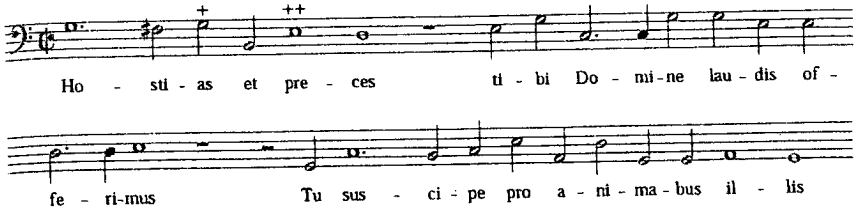


Fig. 4.50b

Of course, the preclassical composers and, later on, the classical and romantic ones selected the moments and pitch where *consequents* "entered" in accordance with the specific building rules of the canon and with the "taste" current at the time. The contemporary composer, liberated from some of the restrictions of this type, creates pages where the concept of canon acquires a new significance, as, for instance, the canon in unison or octave from the 2nd Symphony "Opus Dacicum" by Ştefan Niculescu, based on the *antecedent* shown in Fig. 4.51.

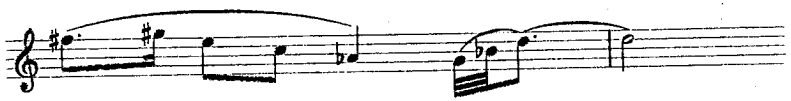



Fig. 4.51

By this canon, the composer aims at a heterophonic effect (2.3.4). He takes some liberties in transforming the theme, for instance (Fig. 4.52):

Fig. 4.52

or, sometimes, because of the sound register of the musical instruments, he transposes some sounds to octave (Fig. 4.53):

Fig. 4.53

Note that these transformations are topological transformations of the type examined in (2.3.1).

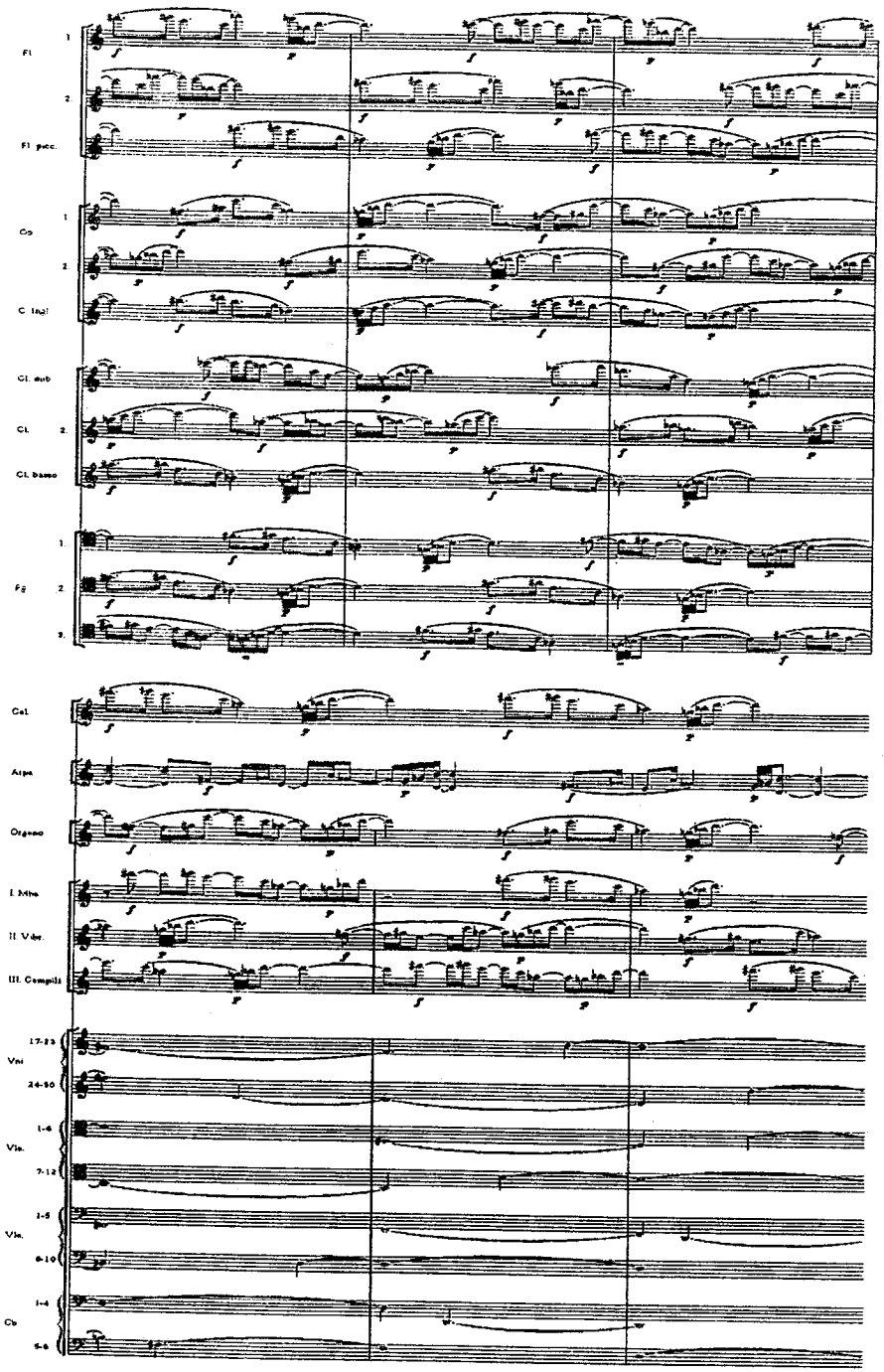



Fig. 4.54

4.4.2. The composer confronted with interpolation problems

Obviously, a certain sound form representing an *antecedent* moving on any trajectory can generate, as mentioned before, a practically infinite family of canons based on the respective *antecedent*. A question arises: can we imagine a generative mechanism which, "loaded" with the *antecedent* of a known canon, could generate a family of canons where the considered canon is present? The problem amounts to defining a trajectory also including the points representing the occurrence moments and pitches of the *antecedent* and *consequents* of the given canon. Consequently, it is necessary to find a function whose graph satisfies these conditions, more exactly, to define a function of the type

$$y = ax^n + bx^{n-1} \ldots vx + z \quad y = f(x)$$

so that the values representing the coordinates of the respective points verify the equation.

This is a problem of *interpolation*; we ask the reader to refer to specific books or computer programs for further information regarding it.

The following paragraph will ilustrate the above exposed ideas.

4.4.3. A polynomial of 4th degree "loaded" with Bernabei's "theme" can generate a large family of canons, that of the Italian composer's being a member of this family

Considering on the *Ox* axis the unit of measure equal to a quarter note and on the *Oy* axis equal to two semitones, the reference point of the form representing the *antecedent* of Bernabei's canon (the abscissa and the ordinate of the first sound at its appearance moment) will have the coordinates (0, –10) and the two *consequents* (16, 4) and (40, 14), respectively.

Making use of interpolation methods, one can imagine an infinity of polynomials representing the second member of an equation of this type (4.4.2) satisfying the equality for the above inserted abscissa values. We suggest the polynomial

$$P(x) = 0.000123\ x^4 - 0.005253\ x^3 - 0.0244\ x^2 + 2.11\ x - 10)$$

$(x = t) \quad y = p$ (pitch)

Here is the trajectory described by the graph of the function on which Bernabei's theme is carried (Fig. 4.55).

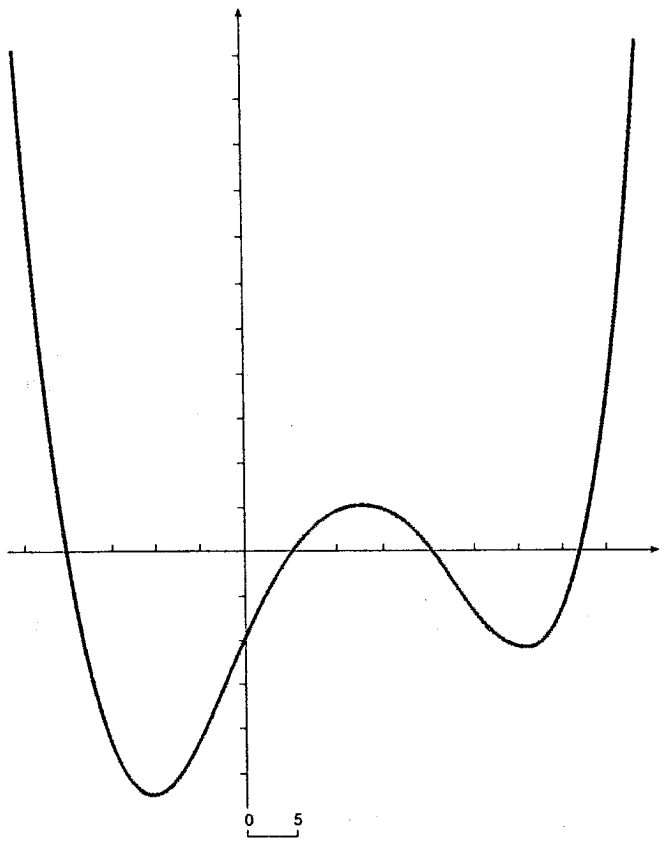


Fig. 4.55

By assigning different values to the polynomial (making sure that the y values are in the real sound zone), the reader can generate as many canons as he wishes, having as *antecedent* Bernabei's theme. By assigning the values 0, 10, and 40 (2.4.1) to the x variable, he *will obtain, as a particular case, Bernabei's canon* (Figs. 4.55).

Making use of a generative mechanism of the above type, the composer could be confronted with the desire to build up canons indicating the pitch at which a certain *consequent* voice should be situated, the way the classicists used to build canons at various pitch intervals (third, fifth, etc.). Should this be the case, the problem arises to solve an equation of the type

$$x = ay^n + by^{n-1} + \ldots vy + z \qquad x = f(y)$$

by assigning values to the y variable.

Operating an equation which exceeds the second degree seems sometimes difficult. This is the reason why it is advisable to work with trajectories consisting of *polygonal lines* which, if need be, may be built so as to include certain given points as well.[10] The polygonal route in Fig. 4.56 also includes points where the *antecedent* and the *consequents* of Bernabei's canon are also situated. (On the Ox axis, the unit of measure is the sixteenth note and on the Oy axis 2 units = 1 semitone.) (Fig. 4.56)

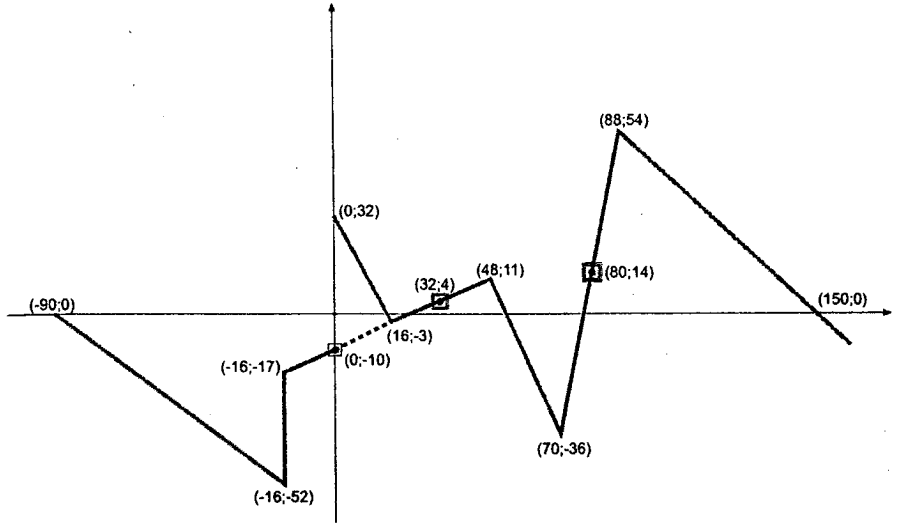


Fig. 4.56

The double canon from Mozart's Requiem *viewed as a particular case of double canons belonging to a set of multi-canons created by a generative mechanism*

A particular case of trajectory that may generate a family of canons (and the simplest to define) is the line crossing the points where the *antecedent* and the *consequents* of a given canon are situated. To the same effect, in the case of the double canon from Mozart's *Requiem* (2.4.1), two trajectories T_1 and T_2 are defined by the couples of points: (8, 2), (24, 7), (16, 2), (32, –5).[11]

[10] The polygonal line is easily built through sequences of lines defined by two given points.

[11] The unit on the Ox axis is equal to the duration of a sixteenth note; on the Oy axis, the unit is 1/2 tone.

Let F_1 and F_2 be the two sound forms representing the double canon antecedents, moving on T_1 and T_2 (Fig. 4.57).

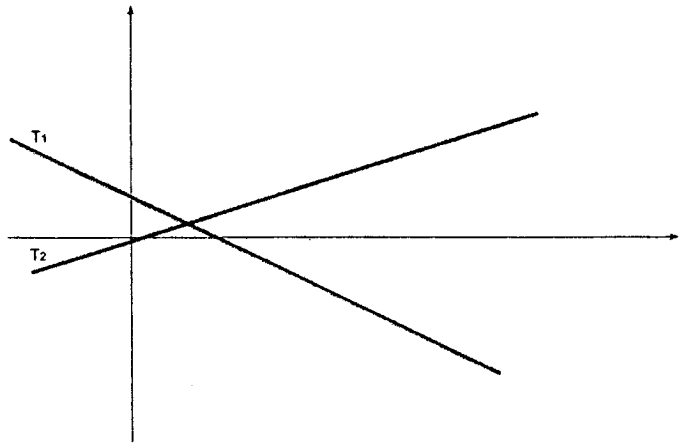


Fig. 4.57

By picking up the positions of F_1 in its motion on T_1 at the moments t = 23, 42, 52, 75 and the positions of F_2 in motion on T_2 at the moments t = 27, 49, 62, 85 and carrying out the union of these images, we obtain in musical notation the following eight voices canon (Fig. 4.58).

4.5. GENERATIVE MECHANISMS MAKING USE OF CONTINUOUS COMPOSED TOPOLOGICAL TRANSFORMATIONS. PARAMETERS VALUES DEPENDING ON TIME *t*. NEW WORKING TOOLS FOR THE CONTEMPORARY COMPOSER

4.5.1. Continuous and discontinuous transformations of a musical sound form

In the preceding section we considered the generative mechanisms as a sound point in motion in sound space $\mathbf{R}^2$, its trace generating a sound; if the point is the reference of a sound form in motion, its trace will be a sound event of the type mentioned in (4.3.1). The motion of a sound point or of a sound form in motion can be viewed as being their transformation through

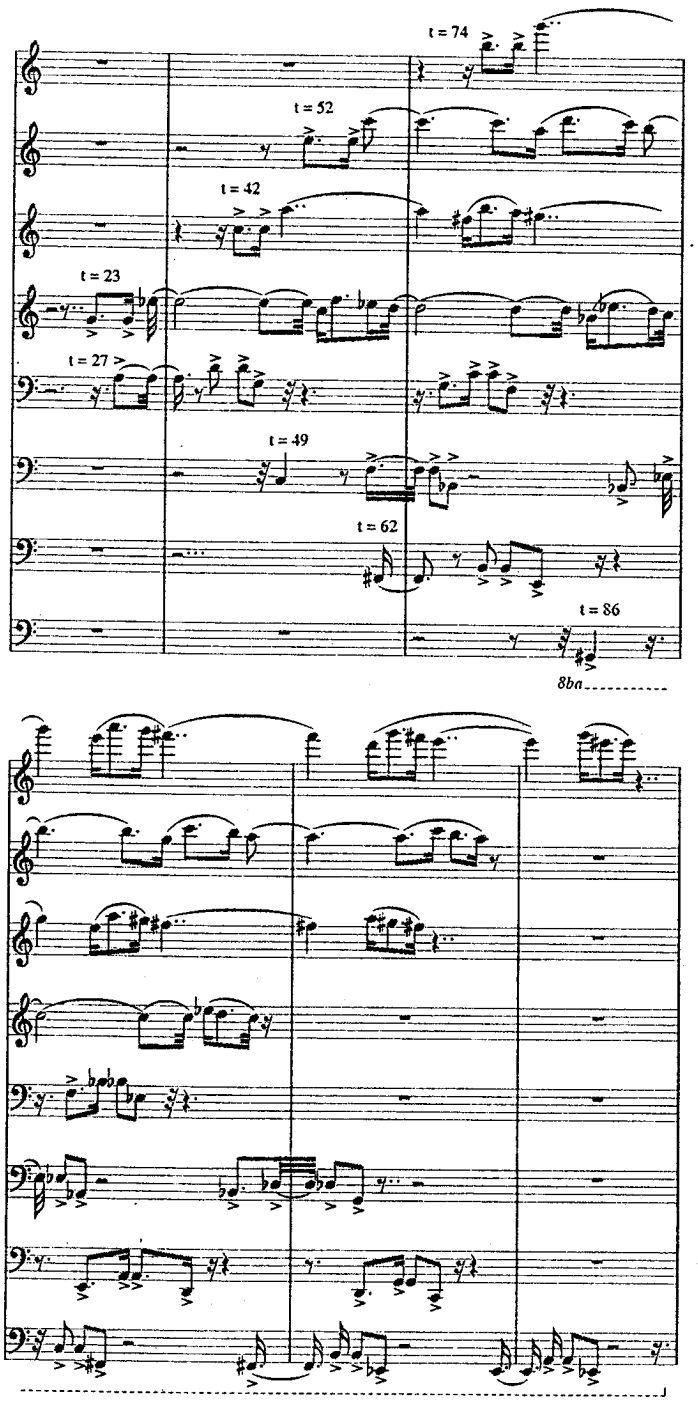



Fig. 4.58

translation (4.2.3). Since the motion is without jumps or interruptions, its evolution in sound space $\mathbf{R}^2$ is *continuous* and can be surveyed in terms of time t. With the mechanisms examined in (4.2.5; 4.2.6) on various time intervals, various transformations by homology, *discontinuously* acting, were associated to the sound trajectory described by the sound point in motion.

4.5.2. Compound transformations. Parameters taking, depending on time t, continuous values in a given values interval

A sound form transformed through translation in sound space may by simultaneously transformed by another transformation. For instance, while travelling on a certain segment of a sound trajectory T, a sound form may also be continuously transformed by homothety, the homothety ratio q being expressed as a function of time. More explicitly, as a sound form covers a segment J of T (J being defined by a time interval $[t_a, t_b]$), the q values will cover a $[q_a, q_b]$ interval of ratio values. A very small increase of t on $[t_a, t_b]$ will correspond to a very small increase (or a decrease, respectively) of the q value on $[q_a, q_b]$. In other words, while t continuously traverses $[t_a, t_b]$, q will continuously traverse $[q_a, q_b]$.

Thus, the considered form will appear continuously transformed by *translation composed with homothety*, its various different images displayed within the time interval $[t_a, t_b]$ being subjected to observation and possibly being picked up discontinuously at any tx moment pertaining to $[t_a, t_b]$.

Two numerical examples.

4.5.3. Continuous transfomation through translation composed with homothety. Homothety ratio value expressed in terms of musical time

A family of homeomorphic images may be generated from an initial sound form SF in motion on a trajectory T, SF being simultaneously transformed by homothety from a center C situated in the center of the rectangle representing the space of SF. C represents the reference point for SF also.

A time interval $[t_a, t_b]$ wherein SF moves on T, and an interval of homothety ratio (q) values $[q_a, q_b]$, are defined. While SF traverses $[t_a, t_b]$, q traverses $[q_a, q_b]$. By picking up the image of T at various moments tx, $(a < x < b)$, a corresponding number of homeomorphic images of SF can be obtained.

Let T be defined by the graph of the function $f(x) = x/4 \cdot \cos x/4$ (Fig. 4.59)

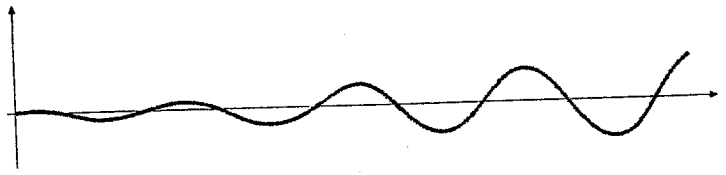


Fig. 4.59

and let SF be a fragment of Brahms's Symphony no. 4 (Fig. 4.60)

Fig. 4.60

defined in the real sound space $\mathbf{R}^2$ by the matrix

0	9	6	9
6	14	8	14
8	7	14	7
14	12	16	12
16	4	18	4
18	0	20	0
20	4	22	4
22	9	24	9

The tables above show:

1. variation of q depending on time t.

t 0 37 68 100

q 1 → 2 → 0.2 → 1

2. the moments (t) when SF is picked up while traveling on T, as well as the values of the reference point coordinates (x_0, y_0) and of the homothety ratio q at these specific moments. We recall that the abscissa x_0 represents musical time ($x_0 = t$).

$t(x_0)$	y_0	q
20	1.4183	1.54054
37	9.10908	2
50	12.4725	1.24516
68	–4.6777	0.2
83	–6.71699	0.575
100	24.7801	1

Figs. 4.61a and b show *SF* (transformed by homothety) at the specified moments and its image in musical notation:

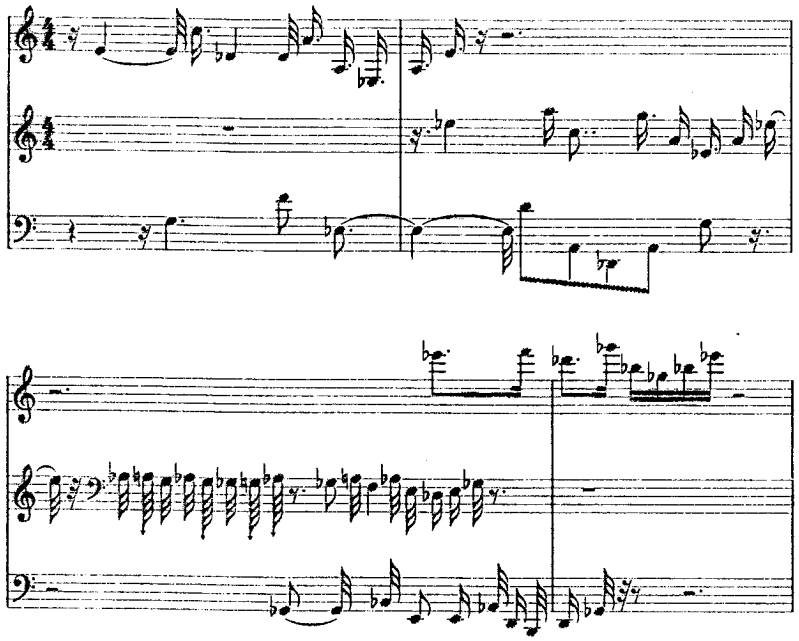



Fig. 4.61a, b

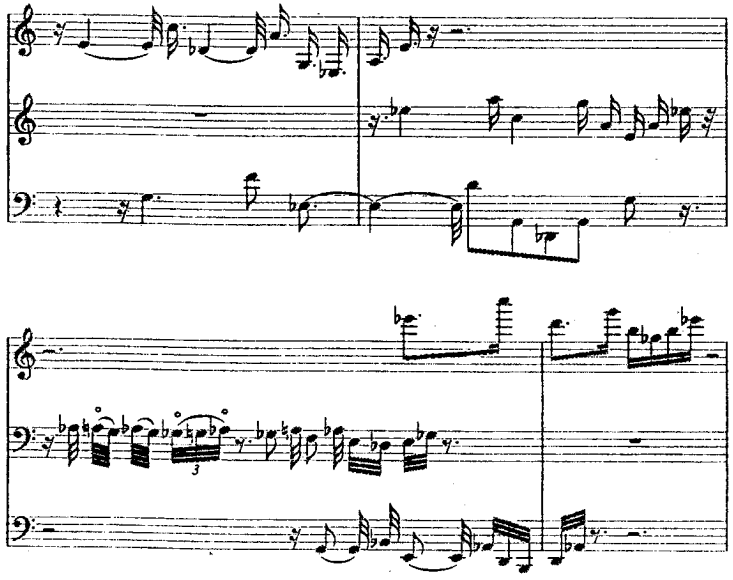



Fig. 4.61c

In order to endow the preceding fragment with a musical significance i.e., in order to avoid the sounds of null duration existing in Fig. 4.61b, in Fig. 4.61c some of the thirty-second notes have been replaced by sixty-fourth notes (the ones marked with a small cercle). See footnote 1 in 4.2.3b.

4.5.4. A sound form continuously transformed through translation compound with homothety and rotation. A rotation angle expressed in terms of musical time

A sound form *SF* in motion on a trajectory may be subject to a continuous transformation through rotation, in the sense that while *SF* traverses a segment of the trajectory within a time interval $[t_a, t_b]$, *SF* continuously rotates, the rotation angle traversing a $[\alpha_1, \alpha_2]$ interval of values.

In the following we will consider the same musical fragment (Fig. 4.60) subjected, while translated on *T*, to a compound transformation of homothety and rotation.

The tables below represent:

1. the variation of the homothety ratio (q) and the rotation angle (α) depending on time (t)

t	0	37	68	100
q	1→	2	→0.2	→1
α	0→	90	→180	→360

2. the moments (t) of picking up the image of *SF* in motion on *T*, the values of the ordonate y_0 (the abscissa x_0 values are implicitly given by t) and the values of q and α at the respective moments.

$t(x_0)$	y_0	q	α
0	0	1	0
10	–2.00286	1.27077	24.3243
18	–0.94858	1.43649	43.73378
37	9.109	2	90
45	2.83	1.5355	113.2258
68	–4.6777	0.2	180
75	18.65716	0.375	219.375
94	–1.45477	0 .85	326.25
100	24.78	1	360

The three considered transformations, i.e., translation (T_1), homothety (T_2) and rotation (T_3), constitute a generative mechanism (*GM*)

$T_3 * T_2 * T_1 \leftarrow GM$

Fig. 4.62a represents *SF* subjected to the action of *GM* at the specified moments *t*.

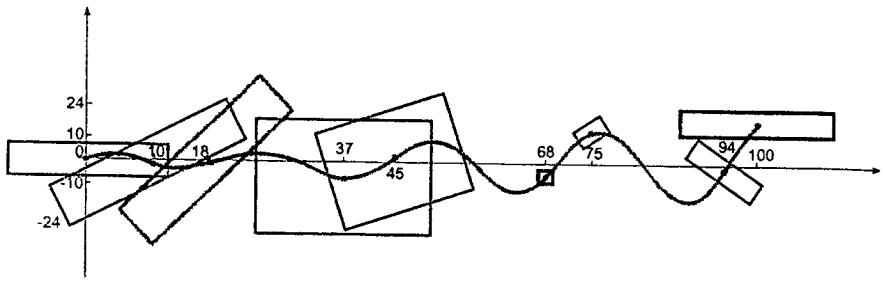


Fig. 4.62a

We recall that when the composer wishes to avoid "glissando" sounds generated by rotation of a sound form, he can apply to the obtained image the *OA* transformation (3.1.3, Ordinate Alignment), thus transforming the glissando sounds into sounds of constant pitch.

Moreover, if the composer wishes to avoid sounds overlap, he can apply to the final image the *AO* transformation (3.1.4, Abscissae Ordering) which transforms simultaneous sounds into concatenated sounds.

The five transformations which *SF* has been subjected to, i.e., T_1, T_2, T_3, *OA*, *AO*, and *MA*, may be composed. Considering the trajectory *T* a component of the transformational system, the sextet $\{T, T_1, T_2, T_3, OA, AO\}$ represents a generative mechanism GM_1.

Let *SF* be subjected to the action of GM_1 and let its image be picked up at moments *t* specified in the above table. Carrying out the *union* of the selected images, we obtain the following musical fragment (Fig. 4.62b).

Again in order to give to the fragment in Fig. 4.62b a musical significance, in Fig. 4.63 the null sounds have been considered thirty-second notes and the value of all other symbols has been augmented multiplying it by 3 (e.g., an eighth note became a dotted quarter note.

Finally, if the composer wishes the obtained fragment to be transformed in a certain musical key, mode or chord structure, he can apply the *MA* (Modal Adjustment, 2.3.9) transformation.

Let us suppose he wishes the fragment mapped in E flat major.

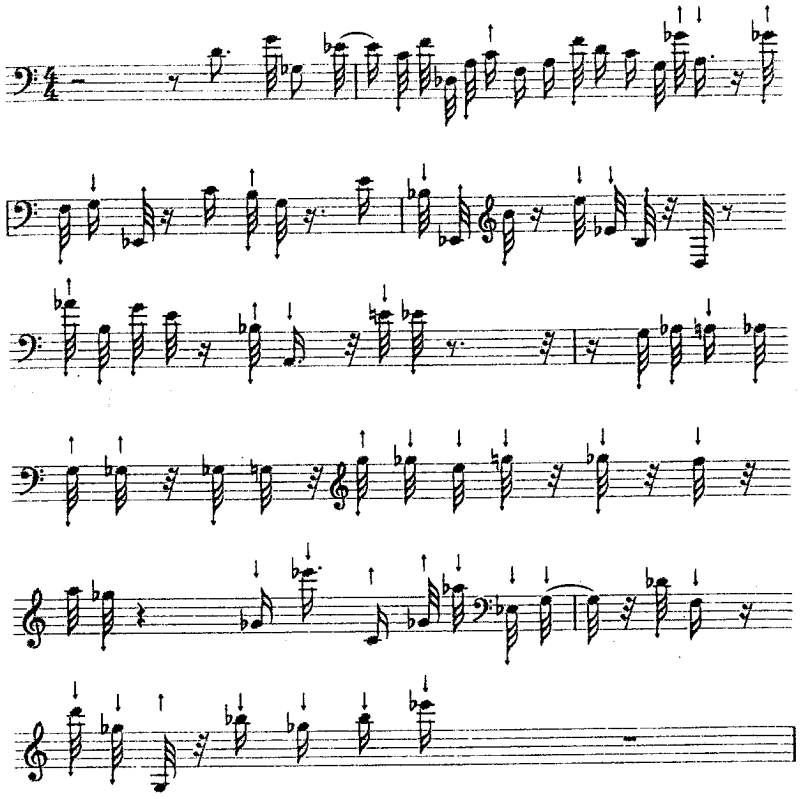



Fig. 4.62b

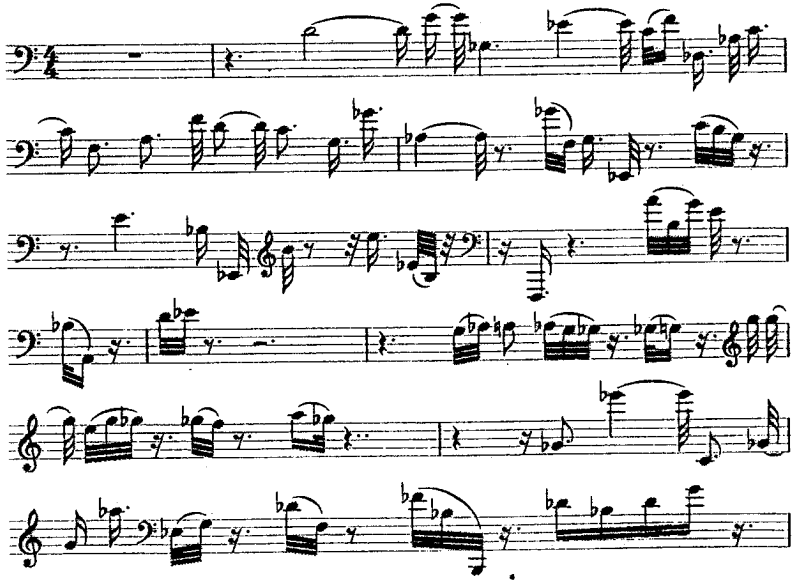



Fig. 4.63

4.5.5. "Sound surfaces" made up of homeomorphic images obtained from an initial sound form

Contemporary music frequently uses "sound surfaces," "manufactured parts," which have to be subsequently assembled in order to obtain some large complex sound structures. The principle of these sound surfaces generated from an initial sound form is not new. It may be found under other denominations, in classical music, in sections constructed by making use of certain methods like: imitations, ostinato basses, canons, developments and variations of a thematic nucleus, etc., fragments which become parts of more ample music structures – fugues, sonatas or symphonies, etc.

While the above-mentioned sections have a perfect building logic, the "sound surfaces" in contemporary music are frequently critical, particularly from the point of view of their inner organisation. The sound structures obtained by generative mechanisms of the previously sugested type allow the "manufacture" of sound surfaces which are defined by the relationships established between the homeomorphic images obtained through the specific procedures generating them. We will refer to such structures as *polyhomeomorphic structures.*

4.5.6. A generating mechanism producing a class of pieces, where a fragment of a piece by Anton Webern may also be found

4.5.6*a.* In contemporary music one may find fragments created by the union of some sets of sounds, each set representing an image obtained through a homeomorphism applied to a given initial form.

In the folowing we will try to prove that the first bars of a piece of the Variations for piano by Anton Webern (Fig. 4.64) have such a structure. Also we will try to create a Generating Mechanism, producing, out of a given sound form, a practically infinite number of images among which those constituting the considered fragment can be found.

It may be noticed that these bars can be considered as being "populated" with a set F of sound forms, each element F_i (i = 1, 2, ...,15) consisting of two sounds S_1 and S_2 each (Fig. 4.65).

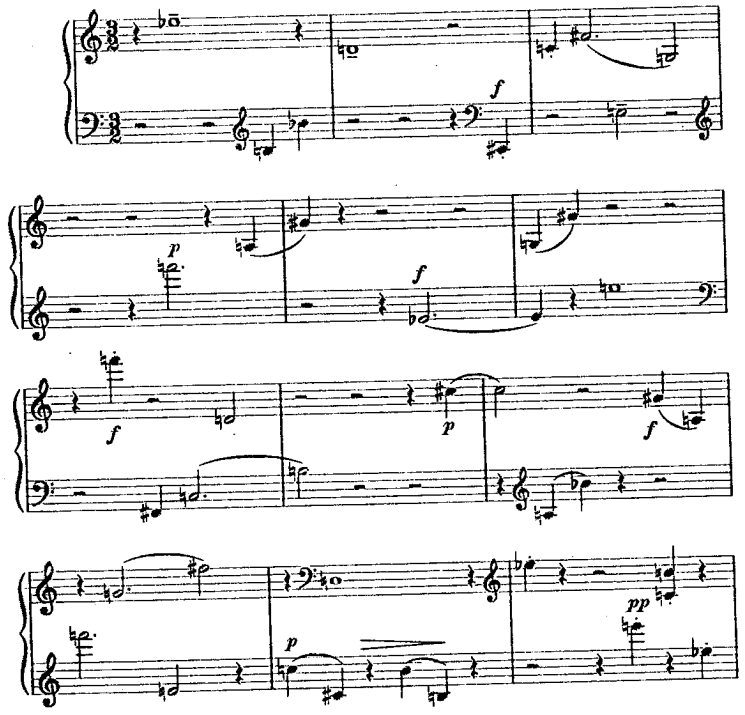



Fig. 4.64

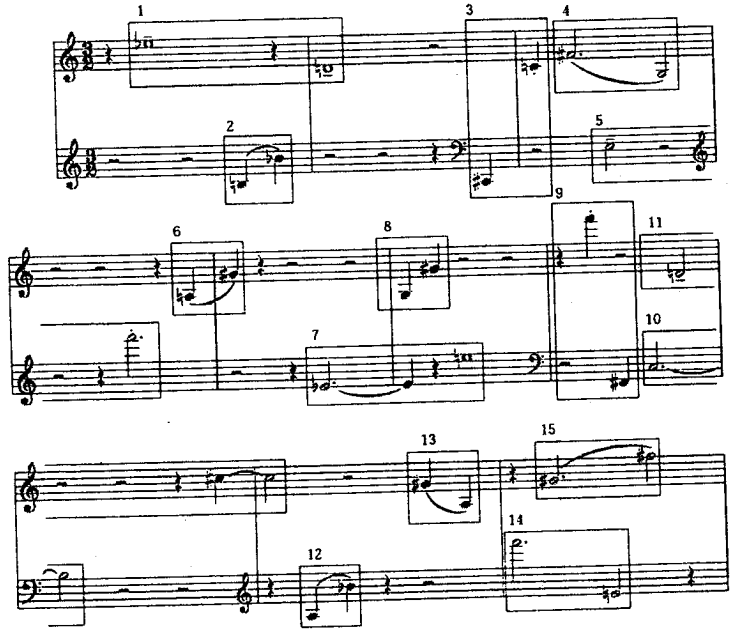


Fig. 4.65

4.5.6*b*. As far as the *pitch* parameter is concerned, the various intervals between S_1 and S_2 are the following:

(a) a large ascending seventh (6, 10, 11, 15)[12] or its enharmonic homologue, the diminished octave (2) (Fig. 4.66a):

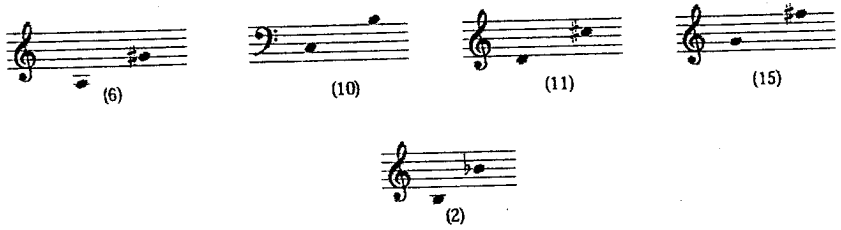


Fig. 4.66a

(b) the inversion of this interval, a large descending seventh (4), (13) (Fig. 4.66b):

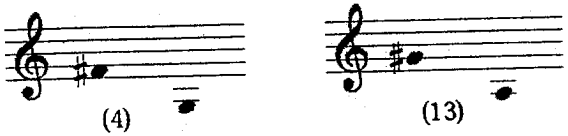



Fig. 4.66b

(c) a small descending ninth (1), an ascending small ninth (12) and its augmented enharmonic homologue, the augmented octave (7, 8) (Fig. 4.66c):

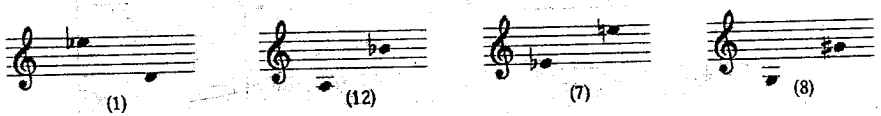


Fig. 4.66c

(d) an ascending triple octave small ninth (5), a descending double octave ninth (14), an ascending diminished double octave (3), a descending diminished triple octave (5) (Fig. 4.66d):

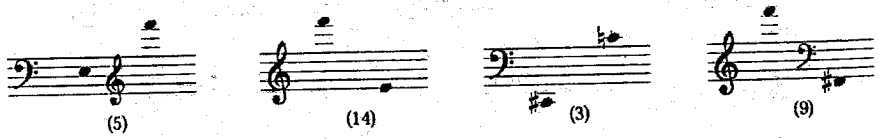


Fig. 4.66d

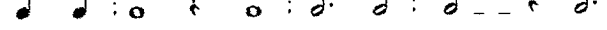
[12] The italic ciphers indicate the respective pitch intervals in Figs. 4.65 and 4.66a,b,c,d.

It is easy to see that all the considered melodic pitch intervals – actually sound forms of two elements – can be interpreted as transpositions, inversions, augmentations, diminutions or compounds of such musical composition methods applied to the first element (6) in Fig. 4.66.

We know that, represented in real sound space $\mathbf{R}^2$, transposition is a translation (2.4.1a) and that reversion, increasing and decreasing of an interval may be obtained through homothety (2.4.2); the set of intervals considering the elements (3), (4) in Fig. 4.68 can be interpreted as consisting of homeomorphic images generated by the proliferation of the element (6) subjected to the action through translation and homothety.

4.5.6*c*. As far as the *time* parameter is concerned, the considered intervals are structured by means of four rhythmic formulae (Fig. 4.67)

Fig. 4.67



constituting four classes of sound forms *A, B, C, D* (Fig. 4.68).

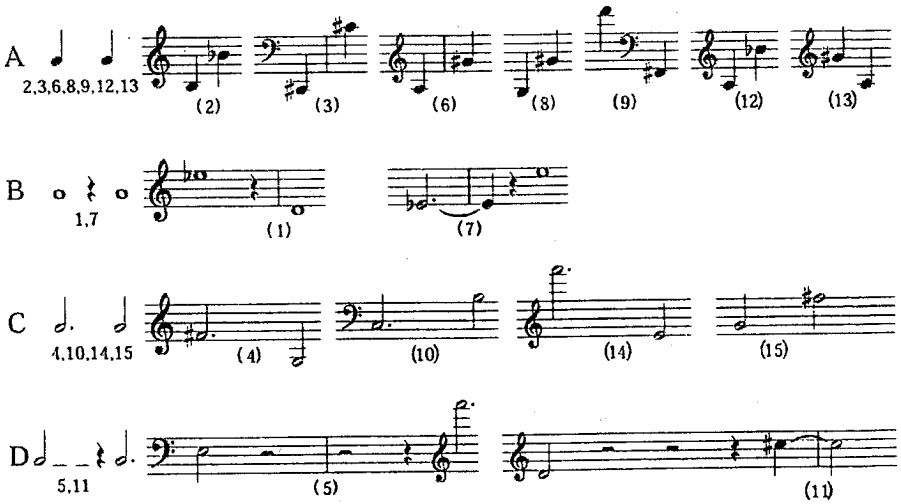



Fig. 4.68

4.5.6*d*. Since the change of a sound duration (*dS*) may be expressed through homothety (2.4.2), and the change of the time interval between two sounds (*L*) may be interpreted as a transformation through translation (2.4.1), the rhythmic structures belonging to the *A, B, C, D* classes may be interpreted as being obtained from one another through topological mappings.

4.5.6e. Considering the reference point (P_r) of the sound forms F (1.1.4) as being situated at the point defined by the occurrence moment (x) and pitch (y) of the first sound of the form (S_1), we may "arrange" Webern's forms in the sound space $\mathbf{R}^2$ according to the mode he "arranged" them on the staves (Fig. 4.64), i.e., locating P_r in the points $\{M\}$ of coordinates x, y: x, y = (8, 15); (32, –1); (88, –23); (104, 6); (112, –8); (184, –3); (216, 3); (240, –5); (296, 29); (312, –12); (320, 2); (392, –3); (416, 8); (432, 29); (440, 7) (Fig. 4.69).

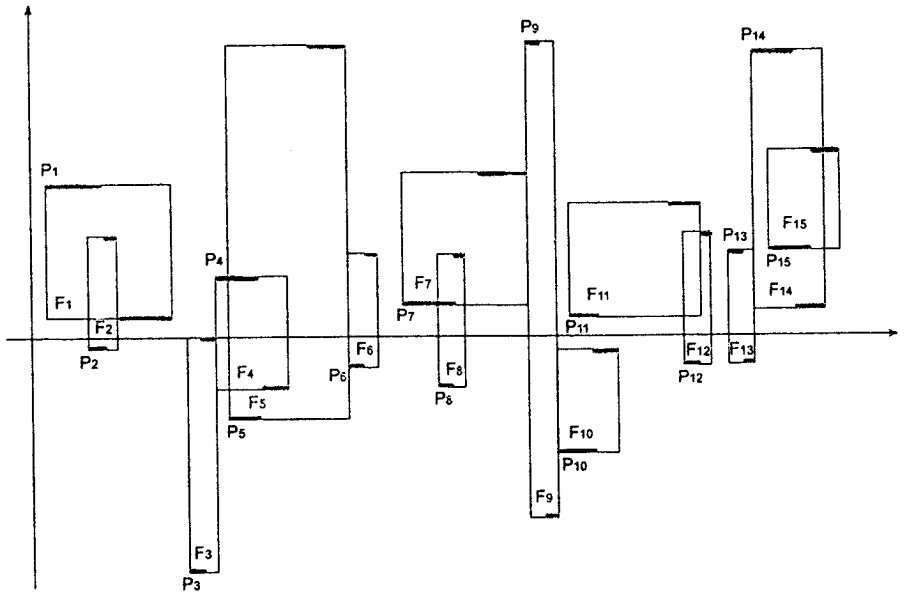


Fig. 4.69

The image in Fig. 4. 69 may be substituted with a table. To each point $P_i \in M$, (i = 1, 2, ... 15) are associated the following data:

1. the coordinates (x, y) of P_r;
2. the value of the pitch interval I in Webern's piece;
3. the duration of S_1 and S_2 (dS_1, dS_2);
4. the time interval between S_1 and S_2 (L).

Thus, the image of F in a point P_i may be defined by a vector V, whose components are x, y the coordinates of the point P_i and the values I, dS_1, $dS_2$and L: $V(P_i) = x, y, I, dS_1, dS_2$ and L. As usual, the values x, dS and L are expressed in thirty second durations (1/32); the values y and I are semitones. The "ascending" I intervals are specified with positive values, the

"descending" ones with negative values. For example, an ascending fifth is expressed by 7; a descending fifth, by –7, Fig. 4.70.

P_i	x	y	I	dS_1	dS_2	L
1	8	15	–13	32	32	8
2	32	–1	11	8	8	0
3	88	–23	23	8	8	0
4	104	6	–11	24	16	0
5	112	–8	37	16	24	40
6	184	–3	11	8	8	0
7	216	3	13	32	32	8
8	240	–5	13	8	8	0
9	296	29	–47	8	8	0
10	312	–12	11	24	16	0
11	320	2	11	16	24	40
12	392	–3	13	8	8	0
13	416	–8	–11	8	8	0
14	432	29	–25	24	16	0
15	440	7	11	24	16	0

Fig. 4.70

The table in Fig. 4.70 defines by means of vectors the musical structure in Fig. 4.65.

4.5.6f. Starting from these remarks, we can imagine a generating mechanism producing from a form F a set of homeomorphic images, a set which also includes the constituent elements of the quoted fragment of Webern's piece.

Let GM be this generative mechanism consisting of four transformations, i.e.:

(1) a transformation τ_1 moving the form F on a trajectory T, which also includes the points P to which we add the point P_0 of coordinates x, y = 0, –3. The F motion in the sound space is surveyed by means of the motion of the reference point Pr (4.5.6e) on T;

(2) a transformation τ_2 which changes the value I of the pitch interval between the sounds S_1 and S_2;

(3) a transformation τ_3 changing the duration of dS_1 and dS_2 (τ_{3a} and τ_{3b}), respectively;

(4) a transformation τ_4 changing the value of the time interval L between the sound's S_1 disappearance moment and the sound's S_2 appearance moment.

4.5.6g. While in a given time interval, P_r traverses on T an interval between two points P_i and $P_{(i+1)} \in M$, the values of the variables I, dS_1, dS_2 and L traverse intervals in $\mathbf{R}^2$ included between the values the respective variables have in points P_i and $P_{(i+1)}$.

Since the motion of F on T can be surveyed by means of the projection of Pr on the Ox time axis, the values of variables y, I, dS_1, dS_2 and L may be expressed in terms of time t:

$$y, I, dS_1, dS_2, L = f(t)$$

Consequently, the *image* of F at a given moment t_x may be expressed in terms of t (Fig. 4.71).

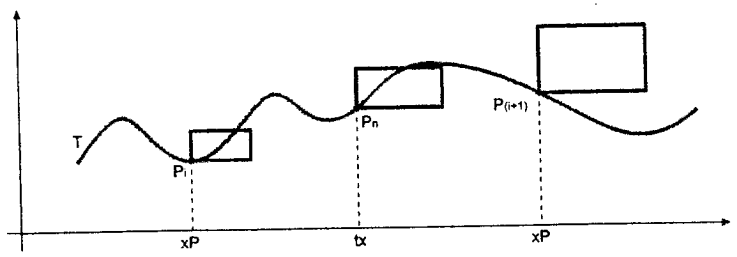


Fig. 4.71

Since in a given time interval, P_r continuously traverses an interval on T, F undergoes *continuously* the effect of the considered transformations.

It can be easily noticed that, while P_r traverses the time interval defined on the Ox axis by the projections of two points P_i, $P_{(i+1)} \in M$, the pitch interval I between the sounds S_1 and S_2, their duration and the time interval L between them increases or decreases proportionally to the time lengths defined by the projection on the time axis of the distance covered by P_r in movement on T.

For example, we observe from this point of view the pitch interval I transformations. Let us consider two consecutive points P_i and $P_{(i+1)}$. Notating by IP_i and $IP_{(i+1)}$ the values of I in the considered points and by $I(P_r)$ the value I in the point P_r of abscissa xP_r, where $xP_i < xP_r < xP_{(i+1)}$ corresponding to the moment tx between the differences $(xP_{(i+1)} - xP_i)$, $(IP_{(i+1)} - IP_i)$, $(xP_r - xP_i)$, $(IP_r - IP_i)$, the following relationships may be set down:

$$[xP_{(i+1)} - xP_i\,] / [IP_{(i+1)} - IP_i\,] = (xP_r - xP_i) / (IP_r - IP_i)$$

wherefrom we determine that at the moment t_x

$$lP_r = \frac{\left(lP_{(i+1)} - lP_i\right)\left(xP_r - xP_i\right)}{xP_{(i+1)} - xP_i} + lP_i$$

Likewise, we may calculate at different moments tx the duration values of S_1 and S_2 as well as the time interval L between sounds, respectively (dS_1, dS_2).

4.5.6h. $$dS_1P_r = \frac{\left(dS_1P_{(i+1)} - dS_1P_i\right)\left(xP_r - xP_i\right)}{xP_{(i+1)} - xP_i} + dS_1P_i$$

and analogously for S_2.

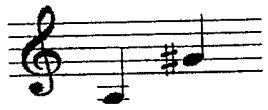
4.5.6i. $$LP_r = \frac{\left(LP_{(i+1)} - lP_i\right)\left(xP_r - xP_i\right)}{xP_{(i+1)} - xP_i} + LP_i$$

4.5.6j. We will further define the generating mechanism GM by making the following operations:

(1) specifiying the trajectory T (4.5.6f):

For the sake of brevity, we have selected as trajectory the polygonal line defined by the points P to which we associate the point P_0 of coordinates x, y (0, –3) whereupon F will start to move on the trajectory T. The points $\{M\}$ will become the flexure points of T. Obviously, the image of many functions may be selected as trajectory on condition that they include the above-mentioned points.

(2) defining the "source" image which will traverse on T:

For our example we have selected the specimen from the class A (4.5.6c), numbered with (6) defined by the matrix MT (Fig. 4.72).

MT

0	–3	8	–3
8	8	16	8

Fig. 4.72

F_0 will start on T from the point P_0 where it is defined by the vector V_0

$$V_0 = 0 \qquad -3 \quad 11 \quad 8 \quad 80$$

In order to study the behaviour of F under the action of GM at various moments $t_1, t_2, \ldots, t_n$, we have to know the action of $\tau_1, \tau_2, \tau_3, \tau_4$ on F at specified moments. Let t_x be one of these moments.

Under the action of τ_1, F_0 will undergo a translation in the sound space on the various segments of the trajectory defined by the points $\{M\} + P_0$.

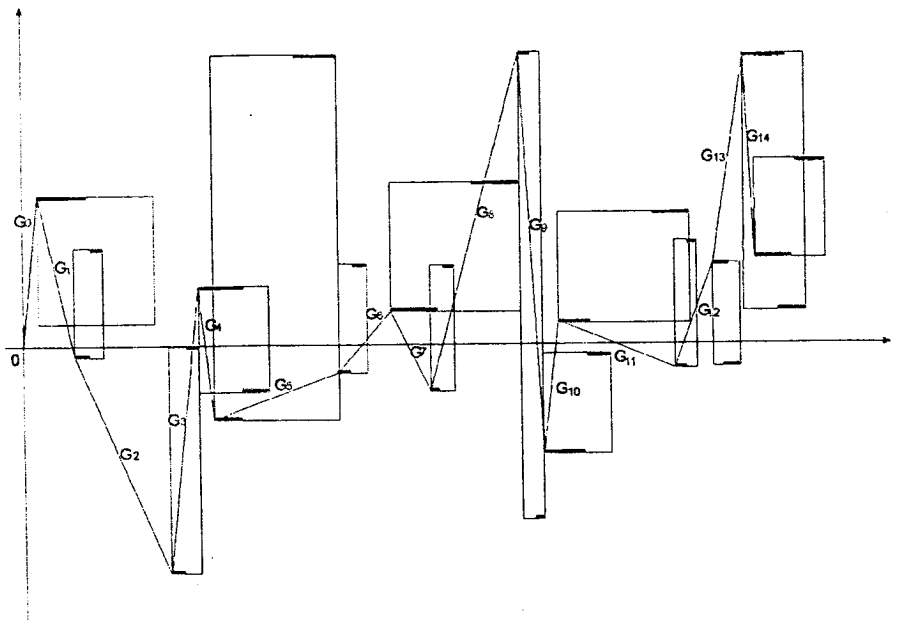


Fig. 4.73

Let z be a segment defined by a couple of points P_i, $P_{(i+1)}$ of coordinates $(x, y)P_i$ and $(x, y)P_{(i+1)}$. The point P_r of abscissa xP_r, $xP_i < xP_r < xP_{(i+1)}$ (Fig. 4.73) will correspond to the moment t_x on T.

The ordinate of point P_r, yP_r is easily obtained from the equation of the straight line including this point, this line being defined by the points P_i and $P_{(i+1)}$:

$$\frac{yP_r - yP_i}{yP_{(i+1)} - yP_i} = \frac{xP_r - xP_i}{xP_{(i+1)} - xP_i}$$

$$\textbf{4.5.6}k. \quad yP_r = \frac{\left(yP_{(i+1)} - yP_i\right)\left(xP_r - xP_i\right)}{xP_{(i+1)} - xP_i} + yP_i$$

The coordinates of the point P_r at the moment t_x, as well as the values of the variables I, dS_1, dS_2 and L in the points P_i and $P_{(i+1)}$ (4.5.6e), being known, it is easy to determine the values of these variables in the point P_r (4.5.6f).

Here is a numerical example:

Let us calculate the values of the vector V components (4.5.6a) defining the image of F at the moment $t = 6$.

Under the action of τ_1, $P_r(F_0)$ will cover T on the segment defined by the points P_0 and P_i of coordinates (0, –3) and (8, 15), respectively. By inserting these values in (4.6.6j), we obtain $y = 10.5$: at the moment $t = 6$, F_0 (i.e., its reference point P_r) will be at the "pitch" $y = 10.5$.

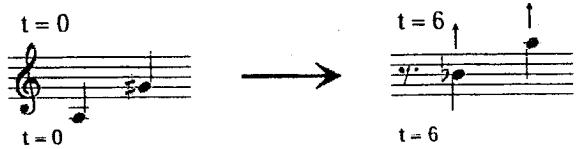


Fig. 4.74

Under the action of τ_2, at $t = 6$, $I = -7$

$$x = t;\ xP_0 = 0;\ xP_1 = 8;\ xP_r = 6;\ IP_0 = 11;\ IP_1 = -13$$

$$IP_r = \frac{(-13-11)\bullet 6}{8} + 11 = -7 \qquad (4.5.6k)$$

Under the action of the compound transformation $\tau_1.\tau_2$

Fig. 4.75

Under the action of τ_{3a}, the duration dS_1 which has the value 8 at $t = 0$ and 32 at $t = 8$, will have the value 18 at $t = 6$.

$$(dS_1P_0 = 8;\ dS_1P_1 = 32;\ dS_1P_r = \frac{(32-8)(6-0)}{8} + 0 = 18 \quad (4.5.6e))$$

dS_1 and dS_2 having equal values both at $t = 0$ and $t = 8$, obviously when $t = 6$, $dS_1 = dS_2 = 18$.

Under the action of the product τ_1, τ_2, τ_3

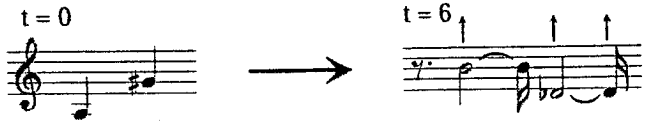


Fig. 4.76

Finally, under the action of τ_4, the time interval between S_1 and S_2 (L) increases from $L = 0$ (value at $t = 0$) to $L = 6$ at $t = 6$

$$(LP_0 = 0;\ LP_1 = 8;\quad LP_r = \frac{(8-0)(6-0)}{(8-0)} = 6 \qquad (4.6.6i))$$

Following the action of the product $\tau_1, \tau_2, \tau_3, \tau_4$

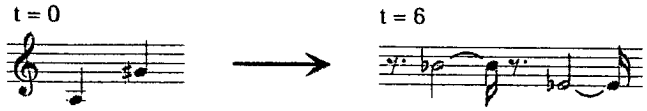


Fig. 4.77

Showing the homeomorphic images obtained through *GM* at the moments $t = 0, 1, \ldots, 8$ (Fig. 4.78) we may follow the *gradual* transformation of F_0 in F_1, i.e., from the source form F_0 to the first homeomorphic form F_1 in the fragment of Webern's piece.

Note that the reversal of F begins around the moment $t = 4$ (Fig. 4.78).

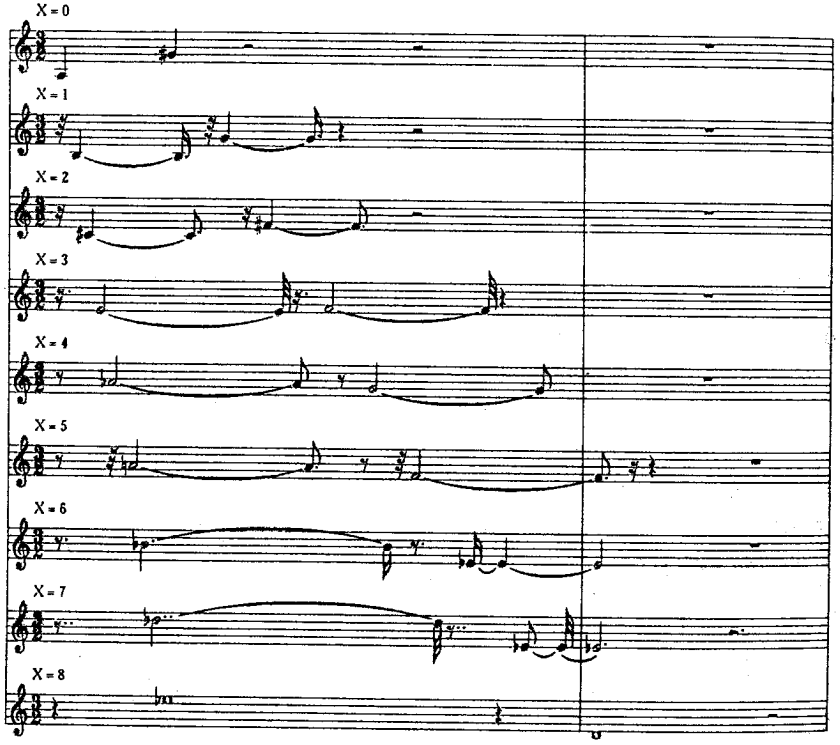


Fig. 4.78

Obviously, when considering values x closer to each other, the transformation will be even "smoother."

Out of the set of homeomorphic images of the power of continuum generated by F_0 through *GM*, subsets of such images may be selected to make up "sound surfaces" peopled with polyhomeomorphic structures.

By the definition of *GM* itself, one of the subsets of homeomorphic images generated by it from F_0 is the page itself representing the bars 1–10 of the piece from the Variations for piano by A. Webern (Fig. 4.79), i.e., the homeomorphic images picked up at the moments t = 8, 32, 88, 104, 112, 184, 216, 240, 296, 312, 328, 392, 416, 432, 440.

In its motion on T, at $t = 8$, F_0 is congruent with F_1; at $t = 32$, congruent with F_2; at $t = 88$, congruent with F_3; at $t = 240$, congruent with F_8, etc. The reader can imagine these transformations considering the image in Fig. 4.69.

The subsets of homeomorphic images may be defined by sequences of abscissae representing the moments when F_0 is picked up while traveling on the trajectory T.

The sequences of abscissae may be provided with certain algebraical structure (e.g., the Fibonacci sequence) or defined at random by a sequence of values included within the interval (0, 440) where *GM* is acting.

The random factor may be restricted according to various criteria, as, for example, selecting a time value for each interval included within the values generating the homeomorphic images in Anton Webern's piece, in other words, the values included among the abscissae of the points defining the polygonal trajectory T.

Let X_1 be the set of these values: X_1 = 3, 19, 67, 97, 106, 141, 213, 227, 263, 305, 317, 369, 407, 421.

Here is in musical notation the image of F_0 "picked up" at the considered abscisae (Fig. 4.79).

In the example in Fig. 4.79, the variables I, S_1, S_2 and L have values within the range given by definition in the points $\{M\}$ (values given by Webern). The operation area of the mechanism *GM* may be extended by associating to some points on the trajectory T (or outside it if T has been modified by new sides and new flexure points) and specifying in these new points to the I, dS_1, dS_2 and L variables other values than those indicated in

the Webernian fragment; hence, different from those in the classes *A*, *B*, *C*, *D* (4.6.6c), Fig. 4.83 is modified associating with new vectors.

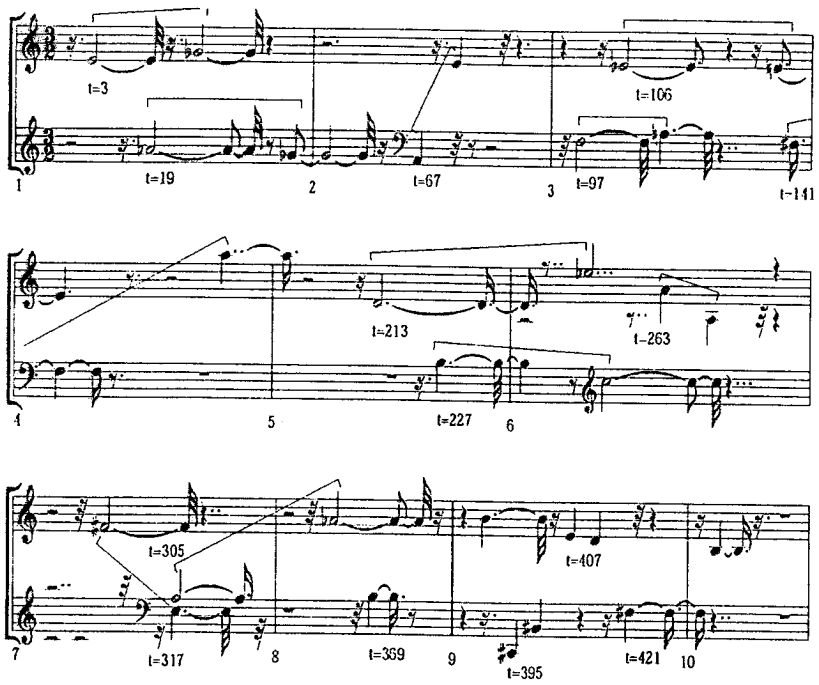



Fig. 4.79

For instance, by inserting between the vectors V_2 and V_3, V_8 and V_9, V_{11} and V_{12}, the vectors V_{2a}, V_{8a}, and V_{11a} (Fig. 4.80)

V	x	y	l	ds_1	ds_2	L
V_{2a}	44	17	–14	4	2	15
V_{8a}	265	11	19	20	6	33
V_{11a}	340	–6	18	4	6	0

Fig. 4.80

and picking up the images at the moments x_2, where x_2 = 4, 15, 38, 70, 90, 120, 144, 200, 255, 280, 301, 325, 345, 400, 420, 422, 432, we obtain the musical structure in Fig. 4.81.

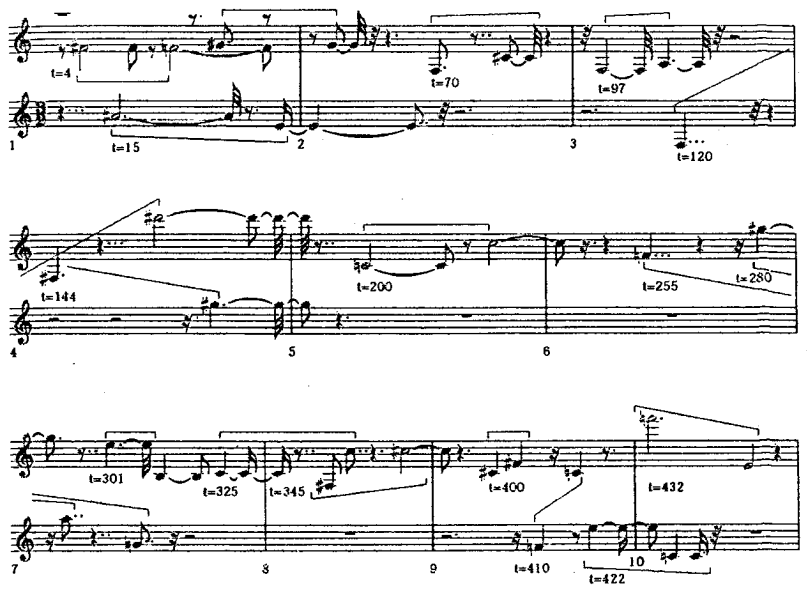



Fig. 4.81

4.6. GENERATIVE MECHANISMS IN VIRTUAL SOUND SPACE R^3

4.6.1. A generative mechanism which, along with the practically infinite set of polyhomeorphic structures it generates, also creates the basic structure of the *Two-Voice Invention* in C Major by J. S. Bach

The analysis of the C major *Two-Voice Invention* by J. S. Bach enables us to note that the piece is generated from a single sound form, from a single musical "motif", to which the great master of polyphony successively applies a series of transformations, i.e., transpositions, imitations, reversions, inversions, augmentations and modal adjustments.

In the chapter regarding the particular cases of topological transformations made use of in traditional music (2.3.1–2.3.9) it has been proved that the above composition methods can be interpreted in sound space $\mathbf{R}^2$:

(1) transpositions and imitations as translations;

(2) reversions as parallel and orthogonal homologies having the homology axis parallel to the time axis and k ratio, $k = -1$;

(3) inversions and augmentations as parallel and orthogonal homologies having the homology axis perpendicular to the time axis and with q values, $q = -1$ (inversion), $q > 1$ (augmentation).

Most of these transformations are followed by modal adjustments *MA* (2.3.9).

Marţian Negrea made a profond analysis of this piece in his treatise "Counterpoint and Fugue." We will make use of it trying to prove that Bach's piece can be almost entirely produced by a generative mechanism processing the basic motif.

This mechanism "acts" through the above-mentioned transformations; and can generate a class, a family of a practically infinite number of such inventions, one of the family members being Bach's short but ingenious piece in C major.

The musical theme – "motif"– forming the soul of this invention is shown in Fig. 4.82.

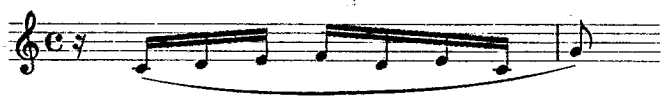


Fig. 4.82

Bach does not always make use of the entire motif; here and there, he divides it into smaller fragments (Fig. 4.83).

Fig. 4.83

Let us call the motif: *M* (Fig. 4.84).

In the first bar the 1st voice starts with *M*, to which the 2nd voice answers with *M* one octave lower. In the second bar, the same thing repeats itself, but transposed a fifth higher. In the third and fourth bars, the 1st voice brings *M* four times in reversed position, accompanied by the 2nd voice repeating three times the *augmented* subdivision *b*. In the fifth bar, *M*, in its original position is once again introduced by the 2nd voice. On beats 3 and 4 of this bar, the 1st voice answers with *M*, but in ts reversed form, accompanied

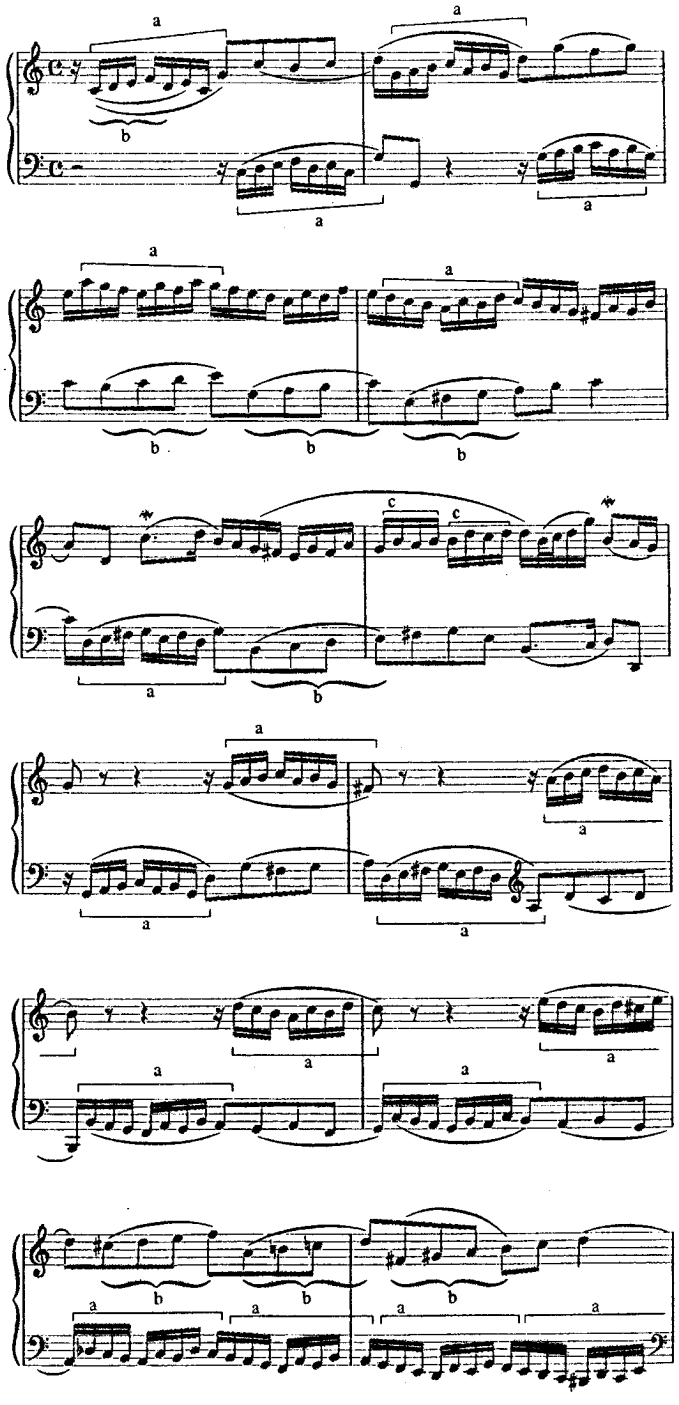



Fig. 4.84

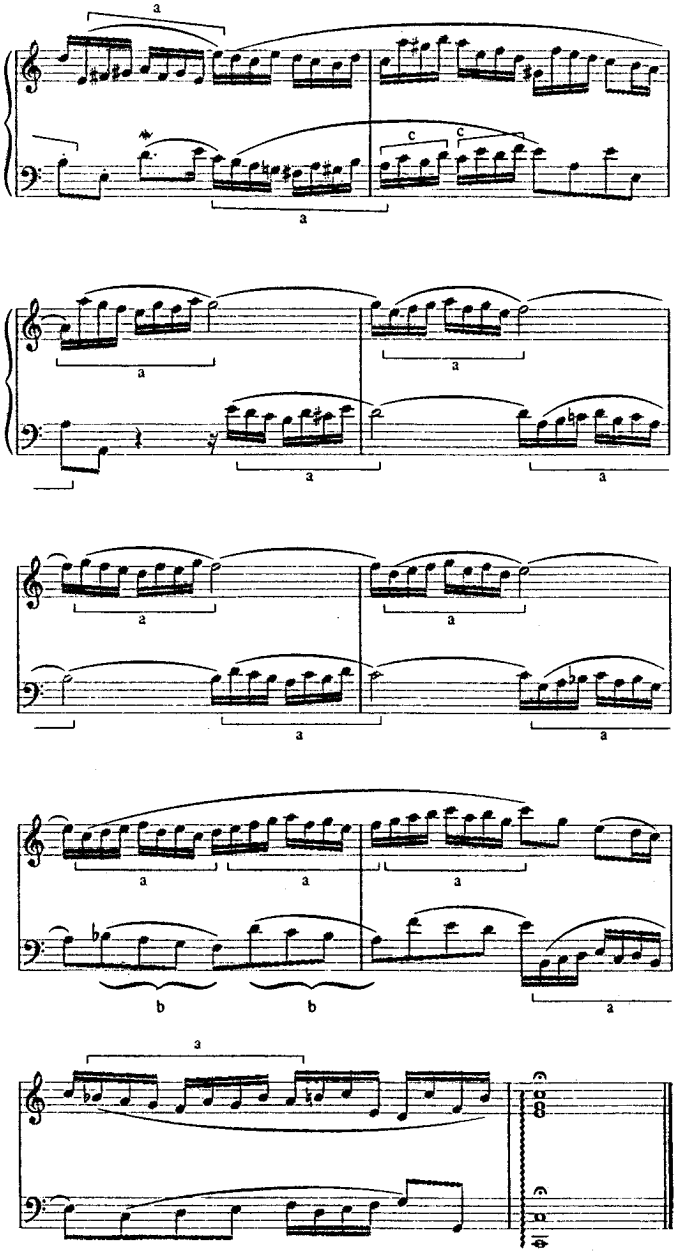


Fig. 4.84

first by the *augmented* subdivision *b*, and then, in the sixth bar by the *augmented* subdivision *c*. In the seventh bar the 2nd voice introduces again *M*, the 1st voice answering it an octave higher. In the eighth bar, the 2nd voice comes in with *M* to which the 1st voice answers a fifth higher. In the ninth bar, on beats 1 and 2, the 2nd voice introduces the inversion of *M*; the 1st voice answers with *M*, in inversed position also. The tenth bar is like the ninth, but a step higher. In the eleventh and twelfth bars, the 2nd voice brings *M* four times in inverted position being accompanied by the 1st voice repeating three times the augmented subdivision *b*. *M* is once more brought in in the thirteenth bar; the 2nd voice answers first with the theme in inversed position, and in the fourteenth bar by the subdivision *c*.

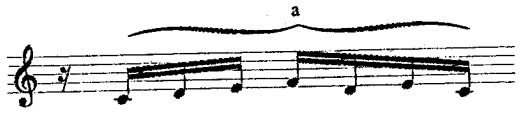
In bars 15, 16, 17 and 18, *M* in its original and inversed position is eight times present. In bars 19 and 20 *M* is accompanied by the augmented *b* subdivision in inversed position and in the twenty-first bar, *M*, in reversed position is once more brought in by the 1st voice.

Here is, in a few words, the structure of this invention, which proves in strong evidence what may be done with a melodic motif consisting of some sounds only.

4.6.2. Bach's "motif" subjected to the action of generative mechanism

In the following we will consider Bach's motif represented by the sound form BH[13] (Fig. 4.85) defined by the matrix:

x	*y*	*x*	*y*
2	0	4	0
4	2	6	2
6	4	8	4
8	5	10	9
10	2	12	2
12	4	14	4
14	0	16	0

Fig. 4.85

The methods previously utilised to describe some generative mechanisms being known, it may be easy to follow up the trajectory

[13] For the sake of simplicity, in the generative mechanism to be suggested, the last note in fragment *M* is dropped.

described by *BH* in motion and the transformations this sound form is subjected to on various time intervals.

For the sake of brevity, we will draw up the action of the generative mechanism for the first six bars only, which, as a matter of fact, include almost all the transformation methods used by Bach in this invention.

Let $\{P\}$ be a set of points in $\mathbf{R}^2$ of coordinates: x, y = (2, 0), (18, –12), (34, 7), (50, –5), (66, 21), (82, 17), (84, –5), (98, 1), (114, 11), (130, –10), (146, 9), (–154, 12), (148, –13), (154, 12), (162, 16)] (Fig. 4.86).

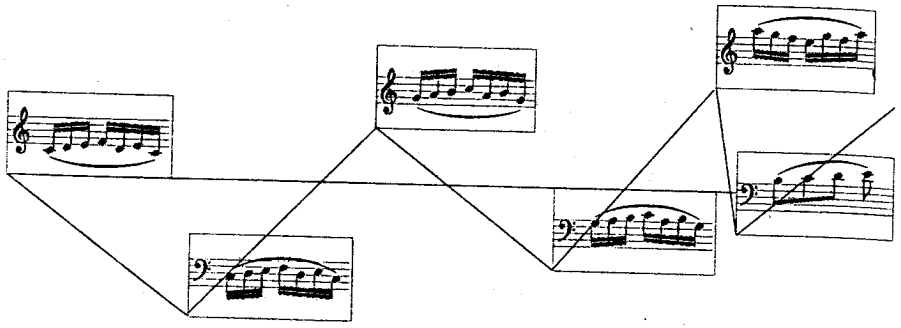


Fig. 4.86

$\{P\}$ represents the positions in $\mathbf{R}^2$ of the reference point P_r of the *BH* and of its image through transformations in Bach's invention. P_r is located in the lower left corner of the sound form rectangle (3.4.2).

Let T be a trajectory defined by the intersection points of the straight lines $D_1, D_2, ..., D_{16}$, defined by a triplet representing the coordinates (x, y) of the elements of $\{P\}$, (taken in the order indicated above) and the value (in degrees) of the angle γ made by the respective straight line with the Ox axis, i.e. D_1: (2, 0, 55); D_2: (18, –12, 106); D_3: (34, 7, 60); D_4: (50, –5, 105); D_5: (66, 21, 80); D_6: (68, –1, 93); D_7: (82, 17, 60); D_8: (84, –5, 100); D_9: (98, 14, 68); D_{10}: (100, –8, 93); D_{11}: (114, 11, 72); D_{12}: (130, –10, 98); D_{13}: (146, 9, 66); D_{14}: (154, 12, 95); D_{15}: (162, 16, 134); D_{16}: (162, 16, 116).

The coordinates (x_1, y_1) of the intersection points are: $D_1 \cap D_2$: (13.6, 16.69); $D_2 \cap D_3$: (19.54, –17.9); $D_3 \cap D_4$: (43.45, 23.37); $D_4 \cap D_5$: (56.47, –33.03); $D_5 \cap D_6$: (67.59, 30.07); $D_6 \cap D_7$: (68.30, –6.72); $D_7 \cap D_8$: (83.16, 19.01); $D_8 \cap D_9$: (84.50, –19.40); $D_9 \cap D_{10}$: (98.74, 15.85); $D_{10} \cap D_{11}$: (101.08, –28.74); $D_{11} \cap D_{12}$: (125.10, 39.03); $D_{12} \cap D_{13}$: (131.80, –22.87); $D_{13} \cap D_{14}$: (146.75, 10.70); $D_{14} \cap D_{15}$: (148.13, –15.58); $D_{15} \cap D_{16}$: (157.02, 26.20); $D_{16} \cap D_{17}$: (186.40, –34.04) (Fig. 4.87).

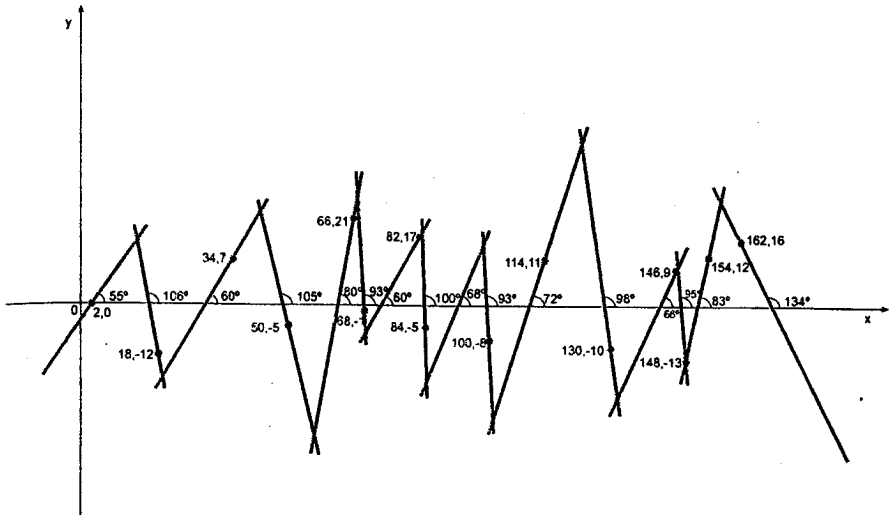


Fig. 4.87

Let I be a number of 15 time intervals defined by the abscissae of the $\{P\}$ elements, i.e.: I_1 [2, 18); I_2 [18, 23); I_3 [23, 50); I_4 [50, 66); I_5 [66, 68); I_6 [68, 82); I_7 [82, 84); I_8 [84, 98); I_9 [98, 100); I_{10} [100, 114); I_{11} [114, 130); I_{12} [130, 146); I_{13} [146, 148); I_{14} [148, 154); I_{15} [154, 162).

Let τ be a number of topological transformations applied to *BH*, i.e.

- τ_1 – translation $P_x \rightarrow P_{(x+1)}$ of the reference point P_r in a plane located in $\mathbf{R}^3$ parallel to $\mathbf{R}^2$;
- τ_2 – transformation through homothety, the homothety center located in the center of the rectangle representing the sound form *BH* and the homothety ratio, $q = f(t)$;
- τ_3 – transformation through parallel and orthogonal homology related to an axis passing through *RP* and parallel to Oy with a homology ratio, $k = f(t)$;
- τ_4 – rotation in the plane including *BH* around a rotation centre coinciding with the point P_r at an angle α, $\alpha = f(t)$ followed by the "reduction to constant pitch" by Ordonate Alignment (OA) and "Abscissae Ordering" (AO) transformations (3.1.3; 3.1.4);
- τ_5 – rotation in $\mathbf{R}^3$ at an angle β, $\beta = f(t)$ around an axis passing through *Pr* and parallel to Ox;
- τ_6 – projection from $\mathbf{R}^3$ in $\mathbf{R}^2$ from a projection center P_c situated over the centre of *BH*, of height $z = f(t)$;
- τ_i – inverse of transformation τ_i, i = (1, 2, ..., 6).

We notice that the parameters k, q, α, β and P_c are expressed in terms of the musical time t, in the sense that, while P_r covers on T a certain segment in a time interval I, the values of these parameters cover an interval between two given values.

In certain cases, the considered time interval may be divided into two or several sections, a certain parameter on each section of the interval might take increasing or decreasing values within two given values. For instance, on a given time interval $I_x = [t_a, t_b]$, $I_x \in \{I\}$ the homothety ratio q traverses the interval of $[c, (c + c/2)]$ values on the $[t_a, +\frac{t_b - t_a}{2}]$ section of I and, on the $[(t_a + \frac{t_b - t_a}{2}), t_b]$, section of I, q traverses the interval of values of $[(c +$ $c/2), c]$. Thus at both ends of I_x, q will have the value c. In other words, during the first half of I_x, q will increase by $(c/2)$ and will decrease by the same value $(c/2)$ in the second half of I_x.

Another example:

During the time interval $[t_a, t_b]$ on the $t_{a+1}, t_{b-1}]$ section, the height z of the projection center P_c in $\mathbf{R}^3$ (2.6.1), will have the constant value Q, while on the time sections $[t_a, t_{a+1}]$ and $[t_{b-1}, t_b]$ $z \to \infty$.

In the following, we will associate to each element in $\{I\}$ one or several elements from τ so that during each time interval defined by an element from $\{I\}$, the sound form *BH* will be subjected to the action of one transformation or of a compound of transformations defined in τ.

Here are two numerical examples:

(a) By associating with the time interval I_1 (2,18) the compound transformation: $\tau_1 * \tau_2$, while *BH covers* I_1, τ will act upon it by shifting it in the sense of *Oy* axis on the interval [0, 12]; τ_2 will act as follows: during the first half of the interval I_1, i.e. up to the moment $\tau = 10$, q will increase from 1 to 1.5, and during the second half of I_1, q will decrease from 1.5 to 1. This is expressed by the table (Fig. 4.88a):

t	2	10	18
q	1	1.5	1

Fig. 4.88a

(b) By associating with I_8 the compound transformations τ_1, τ_3, τ_4 and τ_5 and by defining that during the interval I_8, $I_8 = [84, 98)$, y: $-5 \to 14$, Q: $2 \to 1$, $\alpha: 0° \to 360°$, and on the segments [84, 85), [85, 97), [97, 98) $\in I_8$, the rotation angle β of τ_5 takes, respectively, the constant values 0, 15 and 0.

The above may be expressed by the table (Fig. 4.88b):

t	y	q	α	β	
84	−5	2	0°	0	
				↓	
				15	$t = 85$
	↓	↓		15	$t = 97$
				↓	
98	14	1	360°	0	

Fig. 4.88b

Here is now the diagram of the proposed generative mechanism (Fig. 4.89):

t	q	Q	α	β	P_p x = center of the object z	
2	1	1	0	0	∞	
	↓					
	$2.5t = 10$					
	↓					
18	1	1	0	0	∞	
	↓	↓				
	$3t = 26$	$2.5t = 26$				
	↓	↓				
34	1	1	360	0	∞	
	↓					
	$3t = 42$					
	↓					
50	1	1	0	0	∞	
						$t = 51$
				↓	20	
						$t = 65$
66	1	1	0	180°	∞	
		↓				
	$3t = 67$	↓		↓		
		↓				
68	1	2	0	360°	∞	
						$t = 69$
					20	
						$t = 81$

82	1	0		180°	∞	
	↓					
	$4.5t = 83$					
	↓					
				$20 - t = 83$		
84	2	0		360°	∞	
	↓					
	$1.5t = 93$					
	↓					
					(z)	
98	1	1	360°	180°	∞	
	↓				15	$t = 99$
	$4.5t = 99$					
	↓					
100	1	2	360°	0	∞	
						$t = 101$
					80	
						$t = 113$
114	1	1	0	180°	∞	
	↓					$t = 114$
	$1.4t = 122$			35		
	↓					
						$t = 120$
130	1	1	0	0	∞	
						$t = 131$
					45	
						$t = 145$
146	1	4	0	180°	∞	
		↓				
	$2.7t = 144$				60	
	↓					
148	1	2	360°	0	∞	
	↓					
	$1.7t = 151$					
	↓					
154	1	1	360°	180°	∞	
	↓					$t = 155$
	5.3				–15	
	↓					
162	1	1	0	180°	∞	
						$t = 161$

Fig. 4.89

By picking up the images of *BH* in motion on *T* at $t = 2, 18, 34, 50, 66, 68, 82, 84, 98, 100, 114, 130, 146, 148, 154, 162$ moments (first column in the diagram of the suggested mechanism) we obtain in musical notation the basic structure of the first six bars of Bach's *Invention* (Fig. 4.90).

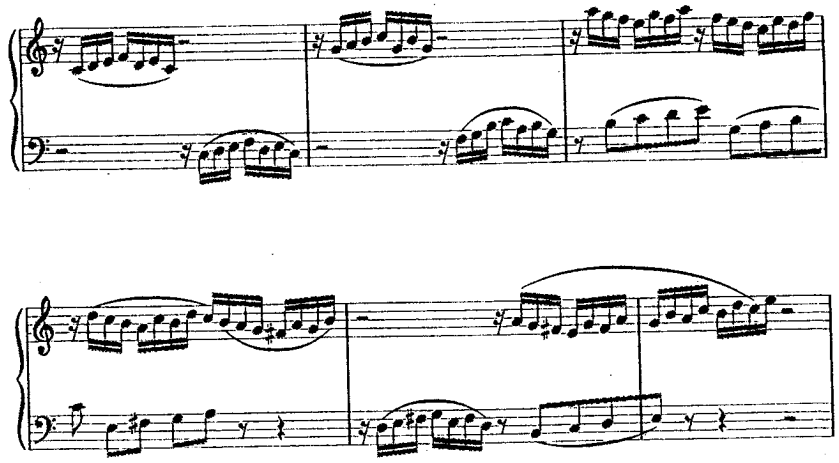



Fig. 4.90

By picking up *BH*'s images at the moments $t = 9, 16, 22, 25, 32, 39, 46, 54, 60, 67, 75, 83, 85, 99, 105, 115, 120, 127, 132, 140, 143, 147$, we have one from the numberless great-great-granddaughters of Bach's *Invention*, manifested, of course, in a contemporary musical language (Fig. 4.91).

4.6.2a: *Bach's* Invention in C Major for Two Voices, *viewed as an element of a set at the power of continuum*

As intended (4.6.2), we have constructed the generative mechanism MG_2 operating only for the first six bars of Bach's *Invention*. Of course, following up the same method, MG_2 can be extended to cover the whole piece. Since the variable t (representing the moments of "observing" *BH*'s image while traveling on *T* under the action of MG_2) can take continuous values on the time axis, it results that the set of "inventions" that can be generated out of *BH* through MG_2 is of the power of continuum, one of the elements of the set being the basic structure of Bach's well known invention in C major (Fig. 4.84).

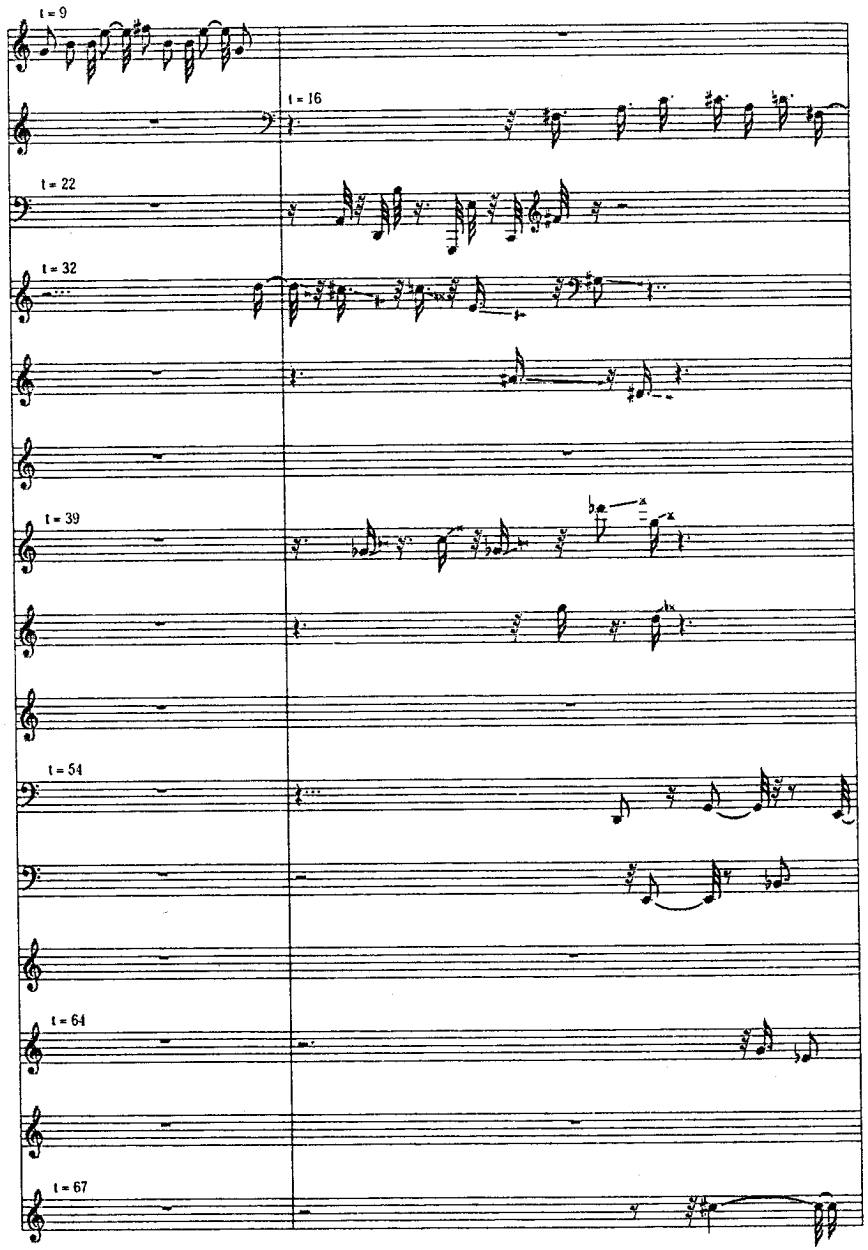



Fig. 4.91

(see also next page)

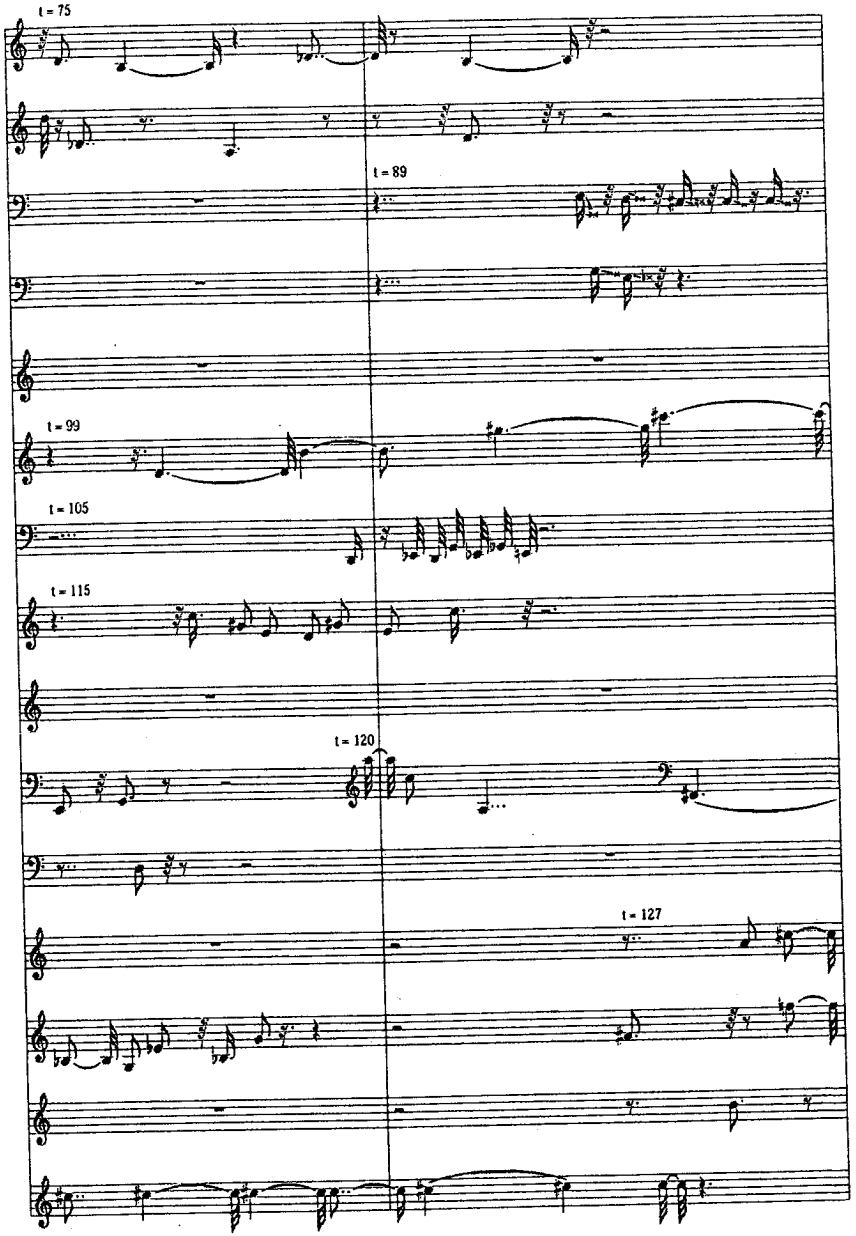



Fig. 4.91

(see also next page)

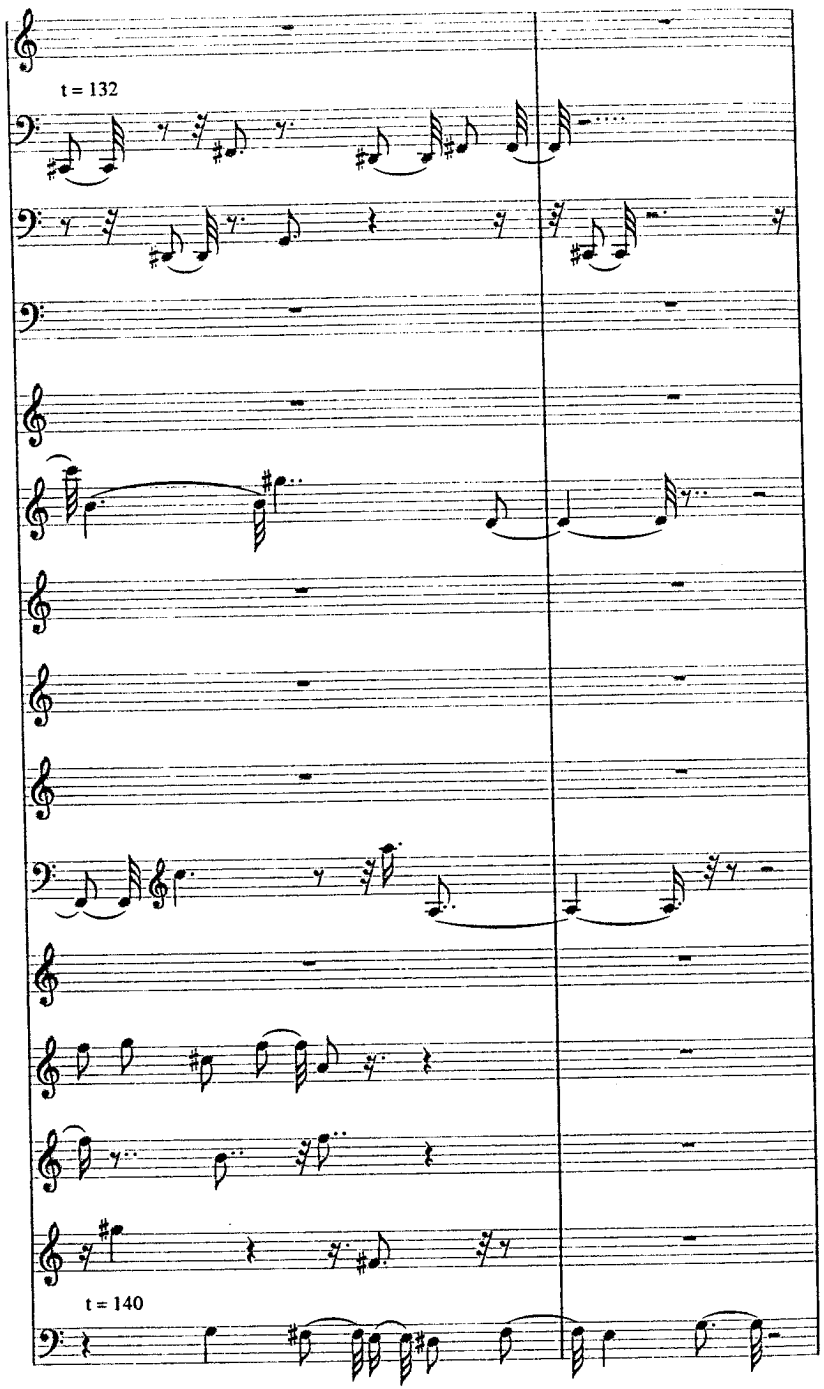



Fig. 4.91

4.7. TWO ASPECTS OF TIME IN GENERATIVE MECHANISMS: CHRONOLOGICAL TIME, MUSICAL TIME

4.7.1. A rotation angle expressed in terms of chronological time

With the preceding generative mechanisms, the various transformation processes were considered as depending on *musical time*, i.e. on the time measured on the *Ox* axis with musical duration symbols (1.5.8).

In the following, we will deal with generative mechanisms where time is considered both under the aspect of *musical time* (measured with musical notation symbols) and under the aspect of *chronological time* (measured with a clocks's unit of measure).

In order to facilitate the understanding of the two different times involved in these mechanisms, we will start with an example.

Let us take a generative mechanism generating a musical structure through rotation of a sound form F in $\mathbf{R}^2$. Operating this type of mechanism, we may observe two steps, i.e.:

(a) F is a geometrical figure in the topological space $\mathbf{R}^2$, the coordinate axes having a purely geometrical significance.

F rotates around a center C_r at a speed v. The value of the rotation angle α is expressed in terms of chronological time ct.

$$\alpha = f(ct)$$

Giving to F a speed of rotation of n degrees per second (*nd*/s), the figure position in $\mathbf{R}^2$ may be "surveyed" at various moments cti of chronological time ($i = 1, 2, ..., n$) (Fig. 4.92).

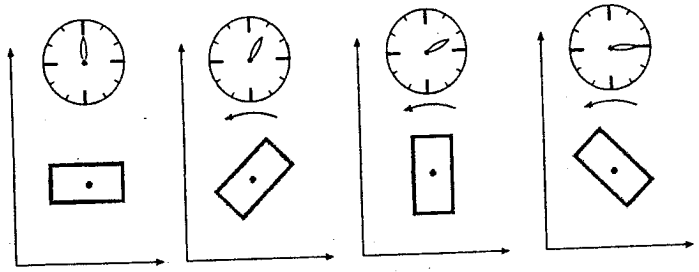


Fig. 4.92

(b) Space $\mathbf{R}^2$ is considered as being a *sound space*, the figure *F* becoming *SF*, a sound form in this space. The space coordinate axes represent *musical time* (*Ox*) and pitch (*Oy*) dimensions.

In (a) *F* is *interpreted as a geometrical being*: its time is the chronological time measured by the clock. In (b) *F*, now considered a sound form *SF*, is *interpreted as a musical being*: its time is musical time, its duration is measured on the time axis *Ox* by musical durations symbols.

It is easy to notice that the *SF* images in $\mathbf{R}^2$ at the *ti* moments depend on the speed of *SF* rotation around C_r.

The speed *v* becomes a parameter of the mechanism.

We invite the reader to imagine this small mechanism operating on the computer screen, himself being able to check the different modes of "populating" a score "surface" depending on the speed of rotation he indicates to a choosen sound form.

4.7.1a: *Speed of a sound form on a trajectory*

Let *SF* be a sound form in motion on a trajectory *T* in $\mathbf{R}^2$. From the point of view of the present study, the speed of *SF*'s projection in the sense of the coordonate axes is relevant (Fig. 4.93).

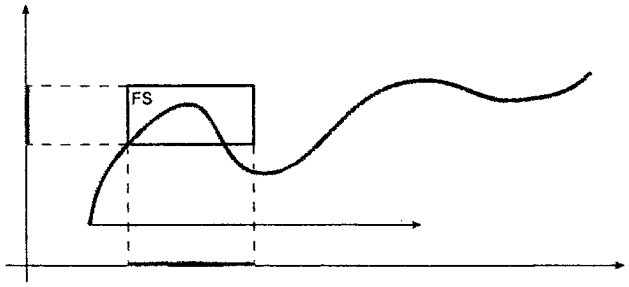


Fig. 4.93

We are particularly interested in knowing the speed of the projection of *SF* on the *Ox* axis, this axis – at this level of operating the mechanism – being considered a pure geometrical one. The speed (*v*) of *SF* projection on *Ox* axis is measured with respect to chronological time (*ct*).

The *SF* images are picked up at different moments ct_x (chronological time)(Fig. 4.94).

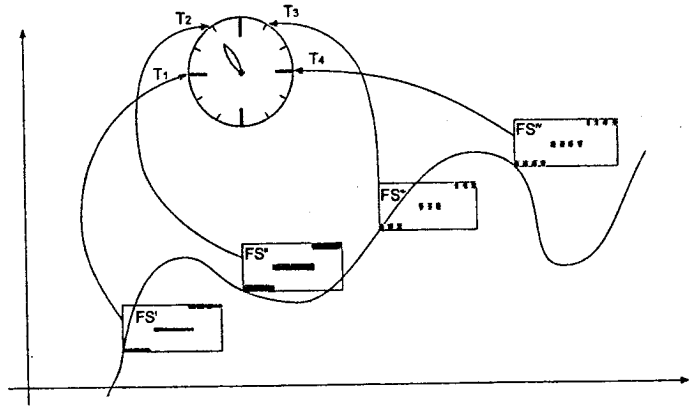


Fig. 4.94

Out of this moment, $\mathbf{R}^2$ is considered as *sound space* and the observed images are interpreted as related to our usual musical significance of the axes: musical time (*Ox*) and pitch (*Oy*) (Fig. 4.95).

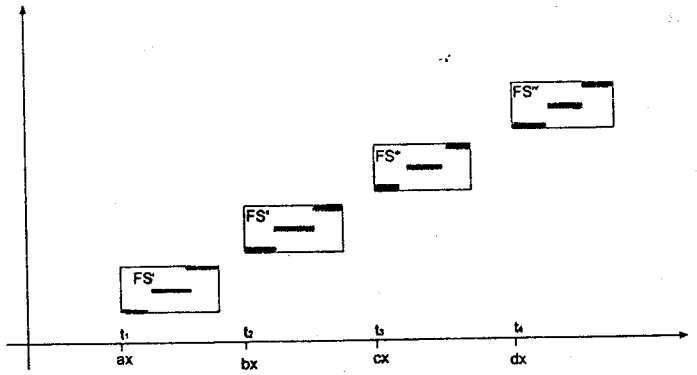


Fig. 4.95

4.7.2. A space with two time reference systems: musical time and chronological time

The image in Fig. 4.109 tries to catch *SF* in motion in $\mathbf{R}^2$, its motion being interpreted, "surveyed" simultaneously on two time reference systems: the clock of chronological time having the second as a unit of measure and the musical time axis having a duration symbol (note) as unit[14] (Fig. 4.96).

[14] In terms of chronological time, the value (duration) of a note depends on the "tempo" of the music, i.e., on the number of "beats" per minute. The time length between

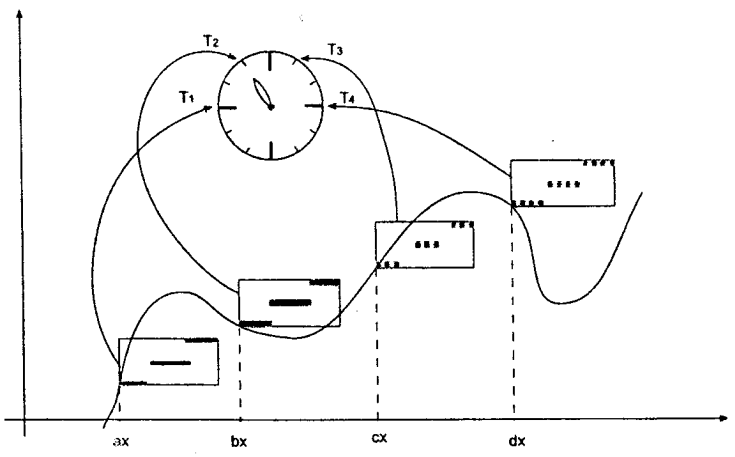


Fig. 4.96

Let us reconsider the mechanism described in 4.3.5 which, depending on the form's duration and on the time interval between picking up moments, generates canons, concatenations or imitations. We recall the fact that in the mentioned case, the position of the various images of an *SF* in mouvement was a function of *musical time* (*mt*) measured on the *Ox* axis:

$$y = f(x), \quad x = mt, \quad y=p \text{ (pitch)}$$

Now we will introduce a generative mechanism which has as time reference system the *chronological time* (*ct*), i.e., the speed (v) of a moving *SF* will be expressed function of chronological time ($y = f(ct)$).

As said before (4.3.5), the speed (v) of *F* "travelling" on a trajectory *T* is considered the speed of *F*'s projection on the *Ox* axis.

Let *L* be – in a geometrical sense – the length (x max–x min) of the projection on *O*x of *F*.

Let $I_{(ct)}$ be a chronological time interval defined by two successive pinpointing moments of *F* moving on *T*. The distance *L* would be covered by *F*'s projection on O*x* in L/v chronological time.

But $L = \Delta$, in other words L represents in sound space the *duration* Δ of the sound form *SF*, duration measured by the musical time unit symbol (thirty-two-note) on the O*x* axis where $x = mt$, musical time. Consequently, we may write that the projection of *FS* on O*x* covers the distance *L* in Δ/v chronological time where Δ represents musical time.

two successive beats defines the duration of the respective note. If and only if note = 60 (note = 1 sec), the two reference systems will have the same time length as unit of measure.

Notating by $I_{(ct)}$ a chronological time interval defined by two successive pinpointing moments of *F* moving on *T*, a generative mechanism of the described type (i.e., having the chronological time as parameter) will generate, like the one described under 4.3.5, *canons, concatenations* and *imitations* on the following conditions:

$I_{(ct)} < \Delta/v$ (canons);
$I_{(ct)} = \Delta/v$ (concatenations);
$I_{(ct)} > \Delta/v$ (imitations).

Since our aim is to obtain the results having the speed v as variable, we may write the above conditions in the form:

$$\text{canons: } v < \frac{\Delta}{l_{(ct)}} \text{ ; concatenations: } v = \frac{\Delta}{l_{(ct)}} \text{ ; imitations: } v > \frac{\Delta}{l_{(ct)}}$$

(Fig. 4.97).

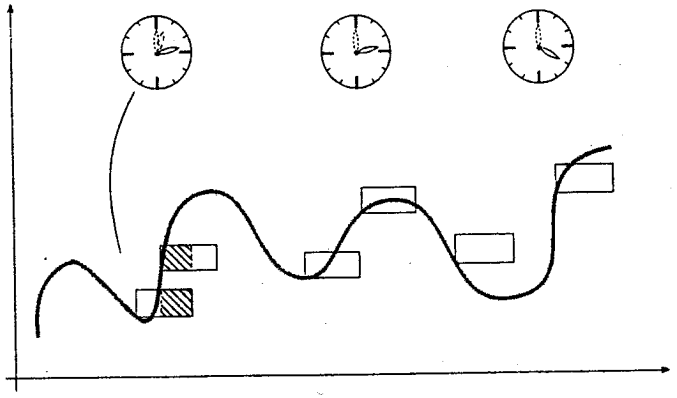


Fig. 4.97

Note that once the position of *F* in geometrical space $\mathbf{R}^2$ is defined by "observing" it at different chronological time intervals, the $\mathbf{R}^2$ space "regains" its property of *sound space* and *SF* its status of musical being defined in terms of *musical time* and pitch coordinates.

The selected images, if desired, can be expressed again in terms of pitch and musical time.

The different characteristics of the two time reference systems are probably even more obvious in the following.

Two mechanisms can be compounded in the sense that during its motion on *T* (at the v_1 speed), *F* makes a rotation also (having as center of

rotation the reference point) at a v_2 speed, (expressed in terms of the number of degrees described by the angle α in a time unit), both motions being observed in terms of chronological time (Fig. 4.98).

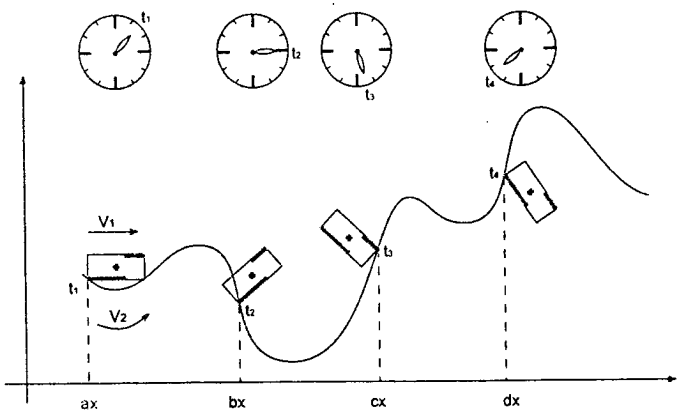


Fig. 4.98

Again, we leave the reader to ponder over the mechanism's way of operating on different trajectories, at various speeds and various moments of surveying the image in motion. We leave the reader to meditate over the music that can thus be generated.

As far as we are concerned, we reserve the right to present him later such a mechanism operating in $\mathbf{R}^3$.

4.7.3. The homothety or homology centres and ratios may also be expressed in terms of the chronological time

Let F be a musical fragment defined numerically by the matrix bellow and expressed in musical notation in Fig. 4.99.

x	y	x	y
76	15	86	15
86	5	98	5

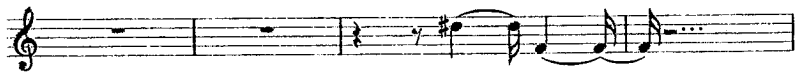


Fig. 4.99

Coordonates (x, y) of the initial homothety center C (87, 6.5).

F moves on a straight line D of equation $56y - 52x + 3848 = 0$, with a speed (in the Ox axis direction) of 12 mm/sec.

The homothety ratio value increases by 0.5/sec.

Moments of pinpointing the sound form F in movement on D: 0.8, 1.2, 1.7, 2.3, 3.1.

Homothety ratio values at the above moments: 1.4, 1.6, 1.85, 2.15, 2.55.

Coordonates (x, y) of the homothety center at the same specified moments: (88.03, 13.03), (91.55, 16.29), (95.94, 20.38), (101.22, 25.28), (108.25, 31.81).

Matrices of the transformed forms (F_a, F_b.....F_e) at the respective moments:

F_a	71.19	15.79	85.19	15.79	F_b	66.67	14.22	62.67	14.22
	85.15	1.79	101.99	1.79		82.67	−1.78	101.87	−1.78
F_c	59.04	10.43	77.54	10.43	F_d	46.99	3.18	68.49	3.18
	77.54	−8.07	99.74	−8.07		68.49	−18.32	94.29	−18.32
F_e	26.00	−11.06	51.50	−11.06					
	51.50	−36.56	62.10	−36.56					

Here is the result in geometrical image (Fig. 4.100a) and in musical notation (Fig. 4.100b):

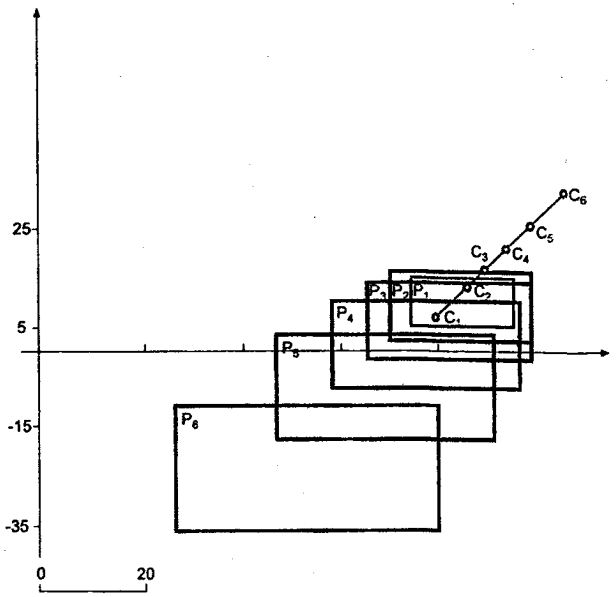


Fig. 4.100a

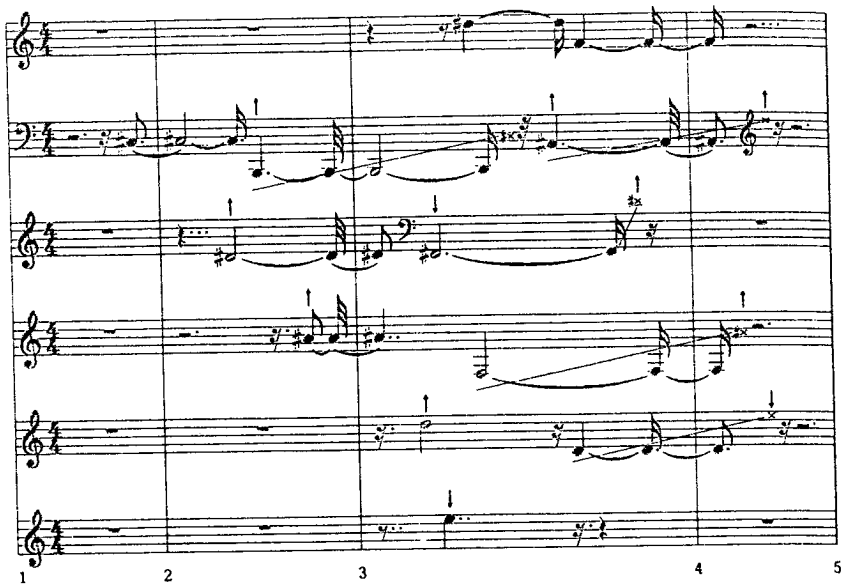



Fig. 4.100b

4.7.4. Dual time (chronological and musical) generative mechanisms in space R^3

(a) The reader will now be able to easily transpose in $\mathbf{R}^3$ transformational processes with parameters whose values are expressed as a function of chronological time.

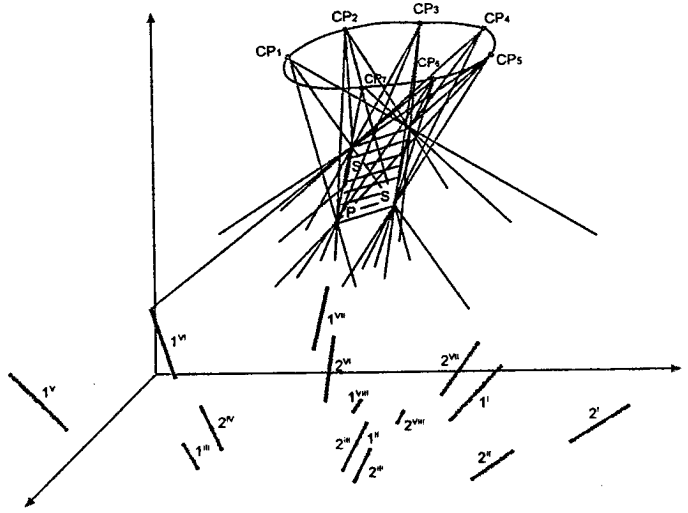


Fig. 4.101

A simple example is the one in Fig. 4.101 where a virtual sound form *SF* is continuously projected in $\mathbf{R}^2$ from a projection centre *Cp* moving on a circular trajectory *T* from $\mathbf{R}^3$ at a speed *v* (Fig. 4.101).

It is obvious that, depending on the shifting speed of C_p on *T*, at different moments C_p will have different positions on *T* and implicitly the moments chosen to "activate" the projection of the virtual sound form *SF'* from C_p on $\mathbf{R}^2$ will generate sound surfaces with different musical meanings (Fig. 102a).

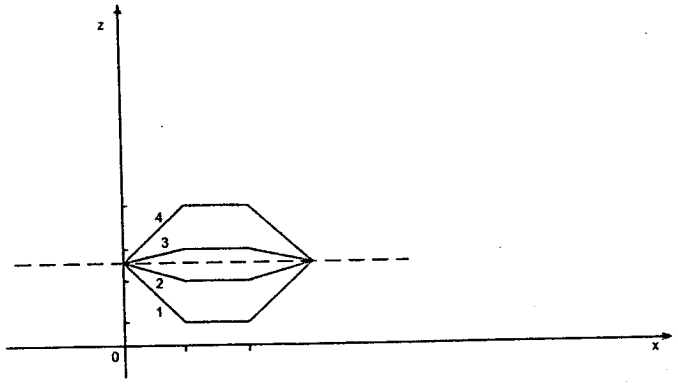


Fig. 4.102a

(b) As a numerical example, let *SF* be a sound form defined by the matrix:

x	*y*	*x*	*y*
0	10	8	3
0	10	8	2
0	10	8	12
0	10	8	17
8	3	16	3
8	8	16	8
8	12	16	12
8	17	16	17
16	3	24	10
16	8	24	10
16	12	24	10
16	17	24	10

In musical notation we have the image shown in Fig. 4.102b.

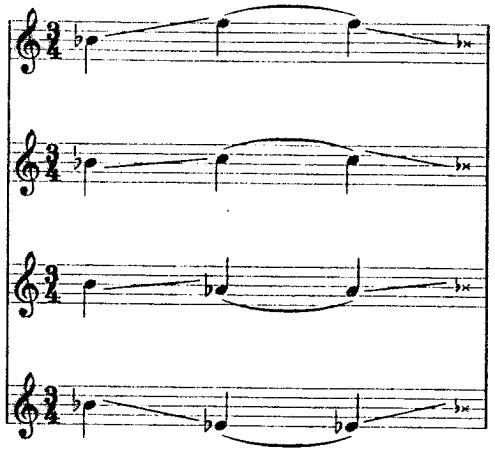


Fig. 4.102b

SF shifts in $\mathbf{R}^3$ on a straight line trajectory *T* included in a parallel plane to $\mathbf{R}^2$, with a coordinate value *z*, $z = 30$. *T* is defined by a point of coordinates *x*, $y = 0, 10$ and an angle α, $\alpha = 0°$ made with the *Ox* axis. The reference point *RP* of *SF* is situated in the center of the left side of the form space *SF* rectangle (3.4.2) ($x = 0$, $y = 10$). *RP* shifts on *T* at a v_1 speed of 24 units *Ox/s.*[15] During its traverse on T_1 *SF* rotates at an angle β around an axis *A* which blends (merges) with *T*. The *SF* speed of rotation around *A*, v_2, $v_2 = 30°/s$.

By projecting *SF* on $\mathbf{R}^2$ by parallel and orthogonal projection lines and pinpointing its position at $t = 1, 2, 3, 4, 5$ ($\beta = 30°, 60°, 90°, 120°, 150°$), we obtain the following concatenated images of *SF* under the action of the compound transformation, translation, rotation and projection (Fig. 4.103a).

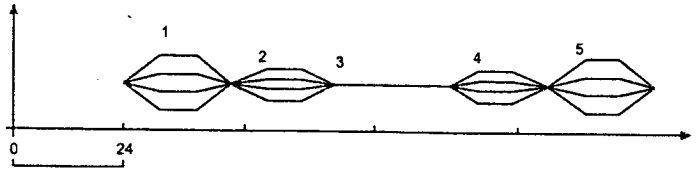


Fig. 4.103a

Because of the rotation, the sounds (1,2,3,4) occurrence of *SF* (viewed from the "bottom" to the "top") may reverse: 1): 1, 2, 3, 4; 2): 1, 2, 3, 4; 3): superposed; 4): 4, 3, 2, 1; 5): 4, 3, 2, 1 (Fig. 4.102a).

Here is this structure in musical notation (Fig. 4.103b).

[15] Lengths unit on *Ox* axis corresponding to the eighth note.

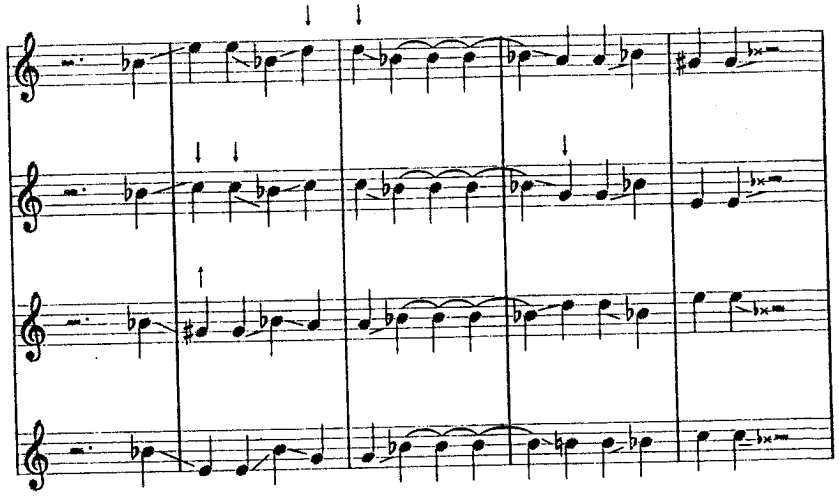



Fig. 4.103b

It may be noticed that, when the form in rotation is perpendicular to $\mathbf{R}^2$, the transformed image no longer keeps the form's topological characteristics exhibiting a degenerated form of the topological form, appearing as a single sound (the entire form is projected on a line).

When the form in motion is pinpointed at $t = 0, 0.5, 1, 1.5, 2$ moments, we have the following "canon" obtained through "telescoping" the images of *SF* (Fig. 4.104a) (rotation angle: 0°, 15°, 30°, 45°, 60°)

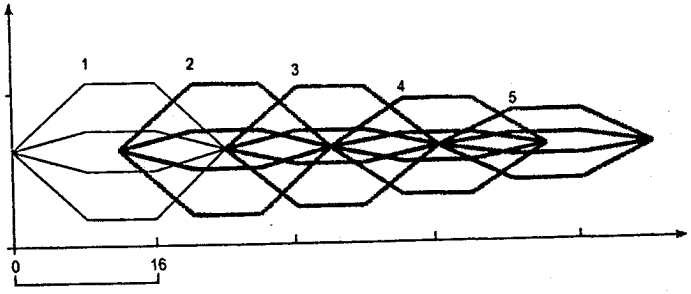


Fig. 4.104a

which, in musical notation, has the following form (Fig. 4.104b):

Here are the "imitations" generated when the form is pinpointed at $t = 3.2, 6.3, 10, 13.4$ moments (β angles = 95°, 139°, 300°, 402°) (Fig. 4.105a)

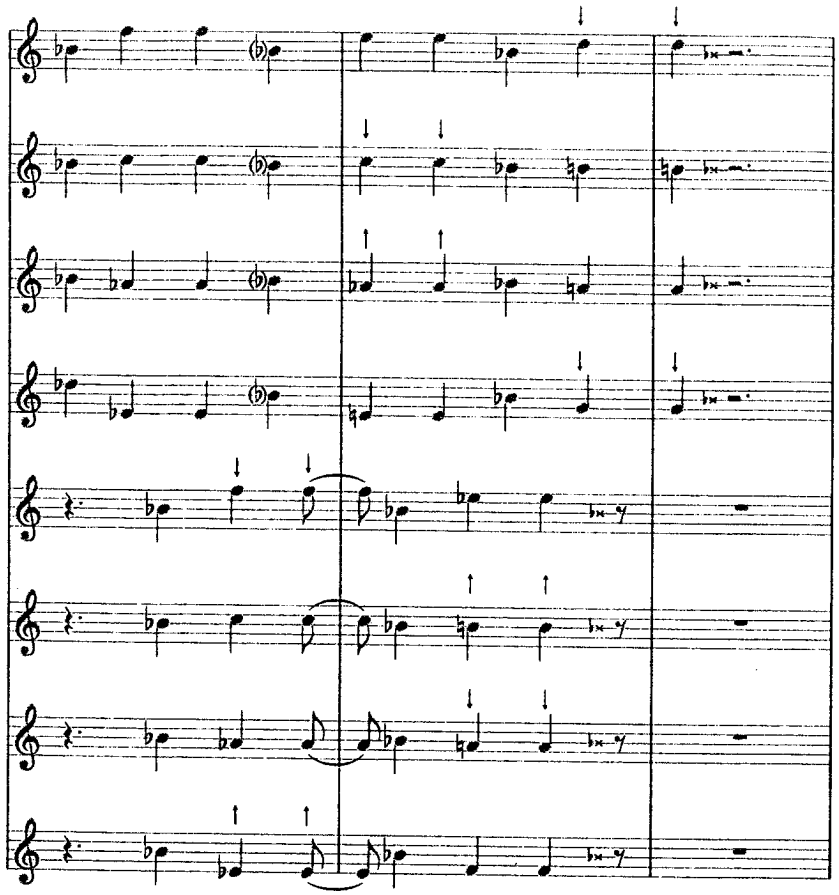



Fig. 4.104b

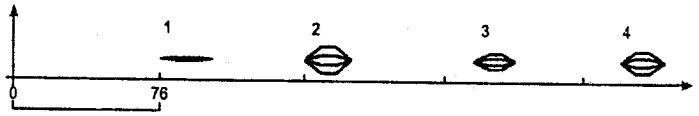


Fig. 4.105a

Sound order 1): 4, 3, 2, 1: 2): 4, 3, 2, 1; 3): 1, 2, 3, 4; 4): 1, 2, 3, 4 and in musical notation (Fig. 105b).

It is obvious that, when changing the values of the variables v_1 and v_2, and using the same pinpointing moments, we will obtain other musical structures.

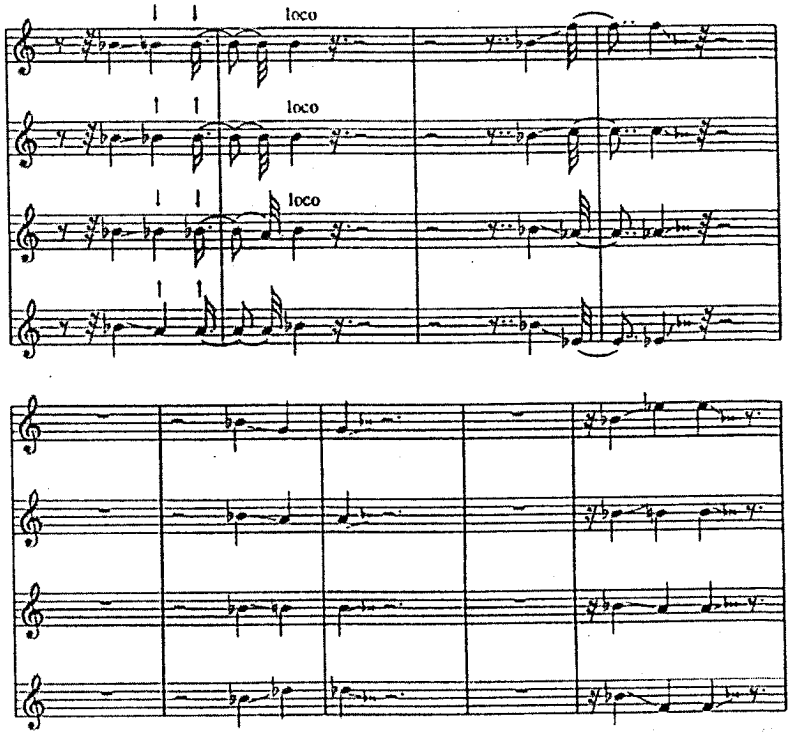



Fig. 4.105b

4.7.5. In order to carry out the small piece "Fantasia on a sinusoid", the composer should remember the integrals

In the following, we will show a generating mechanism where one of the parameters is the distance covered by a sound form on a trajectory represented by the graph of a curve in $\mathbf{R}^3$.

In the virtual sound space of a parallel plane with the real sound space at the height $z = 100$, the reference point of a sound form travels the trajectory defined by the graph of the function $f(x) = 3/50x \sin 3x/50$ (Fig. 4.106)

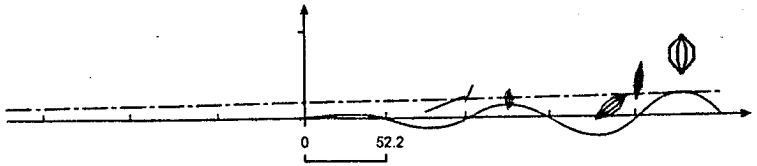


Fig. 4.106

carrying out a complex transformation consisting of:

– rotation in $\mathbf{R}^2$ of an angle α_p, the center of rotation being the form reference point;

– parallel homology having as axis the straight line defined by the reference point of the form and the angle γ made with the axis Ox, the angle of the projecting lines being α_0 and the homology ratio q;

– rotation in $\mathbf{R}^3$ of an angle α_s around the tangent to the curve in the reference point of the form in motion on the trajectory;

– cylindrical and orthogonal projection from $\mathbf{R}^3$ in $\mathbf{R}^2$.

The following conditions are given:

1. At the moment $t = 0$, α_p, γ, α_0, α_s and q have the 1 value.

2. At the moment $t = 10$:

(a) The form traces the third maximum point of the curve (256.8, 14.17).

Let us recall that the length of the curve segment between origin and abscissa point x is calculated as follows:

$$l(x) = 1 + [f'(s)]^2 \, ds = a + (3/50 \sin 3s/50 + s \cos 3s/50)^2 \, ds$$

As a consequence, in order to fulfill the condition **2a**, the form travels on the trajectory at a speed

$$v_0 = l\,(236.8)/10$$

(b) The sound form and homology axis have already made a rotation in plane with $\alpha_p = 90°$ and $\gamma = 90°$, respectively (the homology axis rotating together with the form).

(c) The projections angle (α_0) will have the value of 180°.

(d) The sound form has already made a rotation in $\mathbf{R}^3$ by $\alpha_s = 360°$.

(e) The homology ratio q will have the value 1.

In order to fulfill the conditions required under points **2b, c, d, e,** while the form covers the trajectory, α_p and γ are increased by 9°/s, α_0 by 18°/s, α_s by 36°/s, and q by 0.1/s.

The form transformation in motion on the trajectory may be surveyed related to the time t by picking up the image obtained at different moments t_x.[16]

[16] Allowing for the fact that there is a relation $l(x) = t \cdot vc$ between the time the sound form has used for reaching a point on the curve and the abscissa of this point, the distance $h(x)$ from origin to the abscissa point x may be written as $h(x) - t \cdot vc$. The equation $h(x) = 0$ is numerically solved by the Newton method. We start with the nearest x_0 from the solution and a number of iterations are carried out until the desired accuracy is obtained.

$|x_{k-1} - x_k| < \varepsilon$, ε imposed

$$x_{k+1} = y_k - \frac{h(x_k)}{h'(x_k)} \qquad x_{k+1} = x_k - \frac{h(y_k)}{h'(y_k)}$$

In the following example, we have considered a sound form defined by the matrix and musical notation in Figs. 4.102a, b, this form being treated in various modes under the action of the previously shown mechanisms. The reference point is (0, 10).

Besides the mapping at the moment $t = 10$, the position required by the imposed conditions, a great number of other mappings were carried out from which six have been held, i.e., the images obtained at the moments when the sound form in motion reaches on the curve the neighbourhood of some points of special significance: minimum, maximum or zero value points.

The following table illustrates the values of the various parameters at the respective moments (Fig. 4.107):

No. Remark	t	x	y		s		
1.	0.0941	10	0.339	0.847	1.69	0.09	3.39
2.	1.372	81.5	–4.81	12.34	24.69	0.137	49.39
			Curve first maximum point				
3.	2.234	104.6	–0.05	20.1	40.2	0.223	80.42
			Third zero point				
4.	3.327	133	4.92	29.94	59.89	0.33	119.79
			Second maximum point				
5.	6.2	134.8	–11.04	55.83	111.66	0.62	223.33
			Second minimum point				
6.	7.98	209.2	–0.18	71.84	143.68	0.79	237.37
			Fifth zero point				
7.	10	236.3	14.17	90	180	1	360
			Third maximum point				

Fig. 4.107

Here are, in musical notation, the results forming the small piece named by a commentator "Fantasia on a Sinousoid" (Fig. 4.108 and Fig. 4.109).

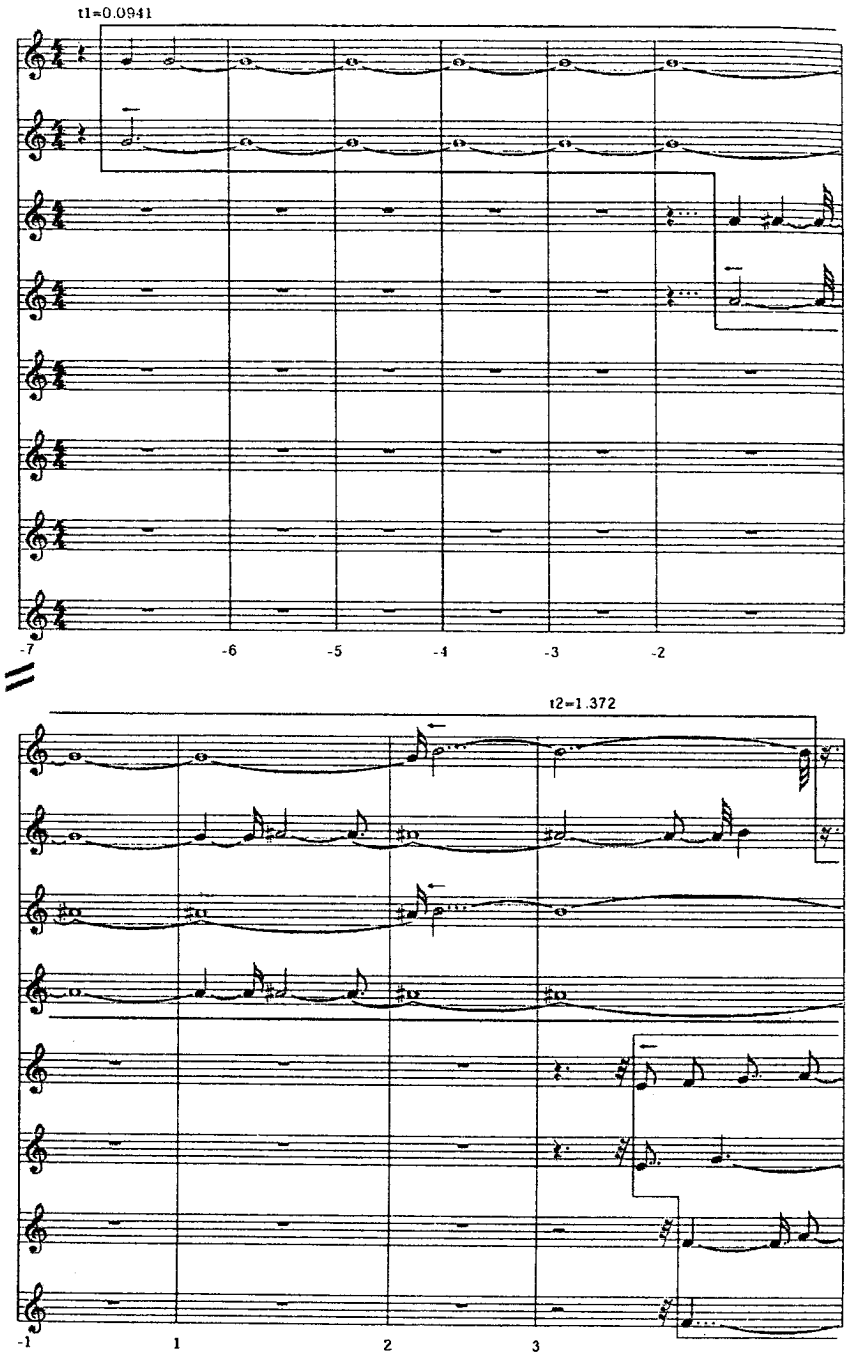


Fig. 4.108

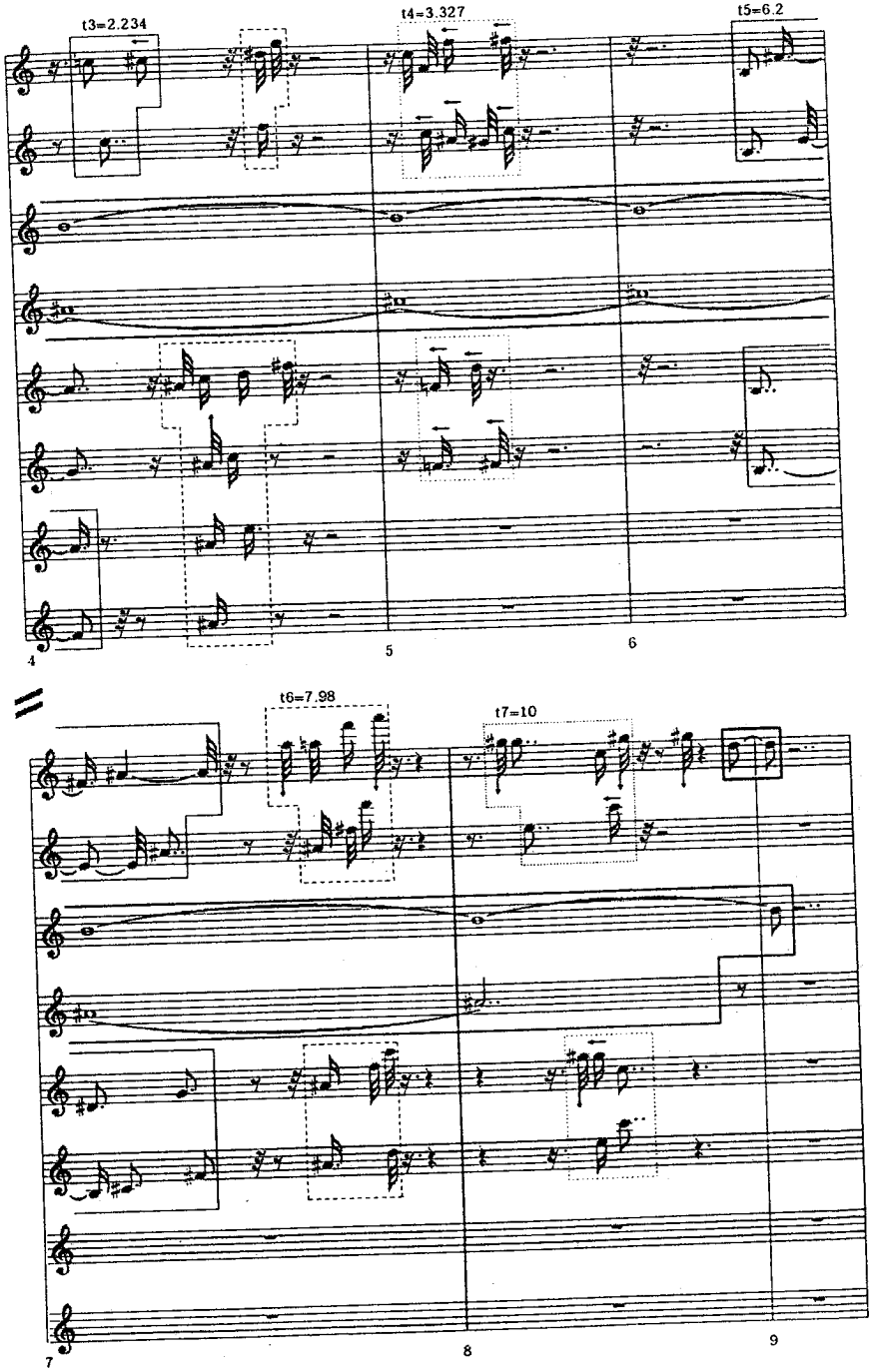



Fig. 4.109

5. ROLE OF COMPUTERS IN GENERATING MUSICAL SOUND STRUCTURES BY MEANS OF TOPOLOGICAL TRANSFORMATIONS AND OF GENERATIVE MECHANISMS

5.1. GENERAL REMARKS

5.1.1. The important amount of calculations needed when dealing with topological transformations applied to sound forms, requires the help of the computer

In achieving transformations and generative mechanisms the computer plays a primordial role. In certain cases of transformations – for instance, translation of a sound form in $\mathbf{R}^2$, which involves calculations easily done "by hand" – the computer is handy, but not essential; yet, in other cases, such as the transformations through homology in $\mathbf{R}^2$ or rotation followed by projections in $\mathbf{R}^3$, the calculations are generally too extensive to be done by hand and, in such cases, the computer becomes indispensable. Thus, the computer substantially extends the range of operations that may be incorporated in the musical composition process.

From the very first we have to remark that, in all the operations connected to the composition process based on topological transformations, the computer is only a *tool*, a means allowing a faster performance of the operations, but it has no kind of *creative* role. Of course, in case of transformations with aleatory variables, the parameter selection can be entrusted to computer; but, here also, the composer may greatly limit the operativity *domain* of the aleatory factor, by putting it, from the creative point of view, under his control.

5.1.2. Basic operations in connection with computer utilisation in sound form transformations

Generally, when dealing with musical compositions based on topological transformations and their generative mechanisms, the computer operation consists of three distinct phases:

(a) entering the sound forms in the computer: in this section the musical forms are referred to as "*objects*";

(b) conversion process;

(c) output of obtained sound form data.

In the following considered drawn up programs, the computer deals only with *rectilinear sounds* in $\mathbf{R}^2$ or $\mathbf{R}^3$; these are input in the computer by the coordinates of the vector defining the considered sound.

The sound occurrence and disappearance data are defined on time axis *Ox*, which is considered as having a random origin; the thirty-second note was selected as a unit of measure. Of course, this duration is relative, since the exact duration in terms of chronological time is defined by the tempo indication for performing the respective fragment. The sound relative durations only are observed in computer operation. The sound pitches are defined on the pitch scale (*Oy*), where the semitone is the unit of measure. The pitch scale may be considered as a bijective mapping of the chromatic scale sounds on the sequence of integers. The pitch axis origin may be selected arbitrarily depending on necessities. More than once, the selection of the C_3 note (central C of the piano keyboard) as origin is the most convenient.

According to the mode of defining the sound forms used in this work, an "object" representing several sounds is inserted as a matrix with n lines and four columns (2.1.3). The values in columns 1 and 3 represent the sound occurrence and disappearance moments. The values in columns 2 and 4 represent the sound pitch at occurrence and disappearance moments.

In the following we display a number of computer programs. Most of them have been used for drawing up the examples in this work.

The first (HOMOTHETY, HOMOLOGY, HOMOPAL, ROTATION, SPACE, COPITCH, SMOOTH) are used to carry out the basic transformations considered in this work. We also display some programs which could be, as we will see, useful to a composer (SHIFT, SPREAD, TRAJECT, MOTION). The last program displayed, PART, carries out the transpositions of digitally defined sound space $\mathbf{R}^2$ in a form close to the musical score.

Of course, the programs may be concatenated, so as to lead to the achievement of some transformation products. For instance, an object, Ob_1, may be subjected to the transformation product Ob_2 PART SMOOTH COPITCH HOMOTHETY ROTATE Ob_1. Let us remember that the transformation product is not a commutative operation: HOMOTHETY ROTATE / ROTATE HOMOTHETY.

In the first phase, the programs were written in APL (Dr. Wilbur LePage at Syracuse University) language for the IBM 360 computer. Their rewriting in BASIC for THOMSON and ATARI computers and in FORTRAN for the CORAL Romanian computer, as well as the logical diagram are due to Otilia Marinescu.

The programs are interactively conceived, most information being required from the user for mapping.

For those who would, possibly, desire to write programs for musical composition, the strictly mathematical aspects of the transformations operated by these programs are exhibited in A3.1, A3.2 (formulae, object behaviour depending on parameter values of a certain transformation, etc.). In the case of some programs we considered to be a little more difficult (HOMOLOGY, HOMOPAL and PART), we have also added logical diagrams that might be used as a guide. Naturally, like in any other range of use of the computer, here also, the programs may be always improved. One can also imagine many other programs for sound form topological transformations.

5.2. BRIEF DESCRIPTION OF SOME PROGRAMS

5.2.1. HOMOLOGY

This program carries out transformations through homology with centre at a finite distance. The user should define the homology axis by the coordinates of a point and the axis angle with *Ox* axis and centre coordinates. The program displays the intervals in **R** where no values of the homology cross ratio may be considered, i.e., values sending to infinite certain points of the object image. (See in this connection the chapter regarding the transformations through homology, A.3.6.)

The transformed matrix is printed. The program may be resumed with another homology ratio.

5.2.2. HOMOPAL

The program carries out transformations through parallel homologies (centre situated *ad infinitum*). The point coordinates and the angle with *Ox* defining the homology axis, the projection angle with axis and the homology ratio are required. When the result is obtained, the program may be resumed with another homology ratio.

5.2.3. HOMOTHETY

The program carries out homothety transformations. The homothety centre and ratio are required. The program allows the introduction for the same operation of several centres and homothety ratios. The results – matrices of 2nd degree representing a family of images through homothety mapping of the object – may be structured in 3rd degree matrix, which may be subsequently considered by programs of the SHIFT- and SPREAD-type we will submit later on.

5.2.4. ROTATION

The rotation creates the image through rotation in plane of an object. A family of images may be obtained in a single operation by specifying several centres and several rotation angles.

5.2.5. SPACE

The SPACE program carries out the sound object transformations in the virtual sound space $\mathbf{R}^3$ and their projection in the real sound space $\mathbf{R}^2$.

The program requires from the user:

a. object positioning in $\mathbf{R}^3$ through a reference point of coordinates *y, x, z* (object centre, lower left corner, middle of left side or any point of the object selected at random);

b. definition of a rotation axis through a coordinate point *x, y, z* and an angle α of the axis with the *Ox* axis;

c. definition of the β angle of object rotation in its plane around the reference point;

d. definition of the γ rotation angle of the object in $\mathbf{R}^3$ round the rotation axis;

e. when the centre is at a finite distance (conical projection), the centre coordinates *x, y, z* are required.

When the centre is to infinity (cylindrical projection), the projection angle with the real sound space $\mathbf{R}^2$ is required.

5.2.6. SMOOTH

This program generates the successive ordering of sound abscissae *AO* (3.1.3), i.e., it deletes sound overlappings. For instance, an object representing a "melody" (successive sounds), may generate through rotation sounds that overlap. SMOOTH deletes them by cutting the sound that occurs first (Fig. 5.1).

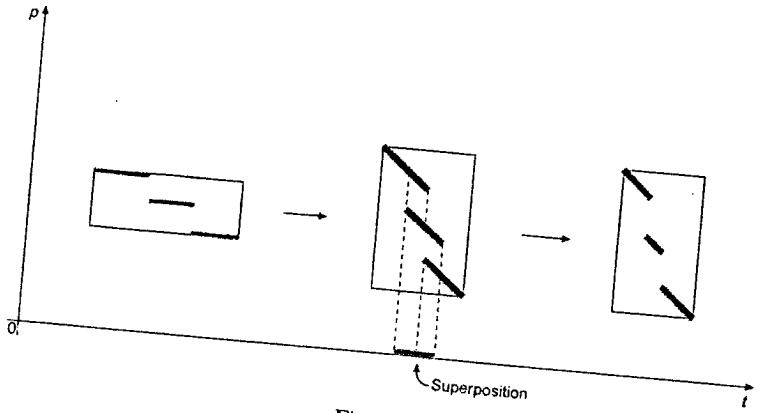


Fig. 5.1

5.2.7. COPITCH

This program is used to align the ordinates (ordinate alignment OA,3.1.3), i.e., it acts on the objects (forms) including "glissando" sounds (2.1.2.) by transforming the same in sounds of constant pitch with an equal value with sound pitch at occurence moment (Fig. 5.2).

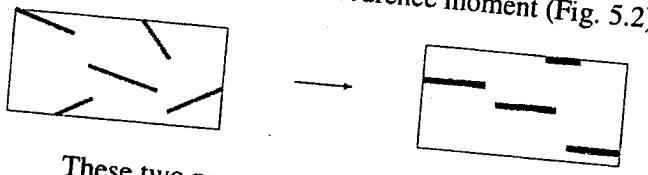


Fig. 5.2

These two programs may be concatenated:

$Ob_2 \leftarrow$ COPITCH SMOOTH Ob_1

5.2.8. MODADJUST

These programs "adjust" the sounds of a sound form, so that the considered sounds belong to a certain mode or key. The three suggested

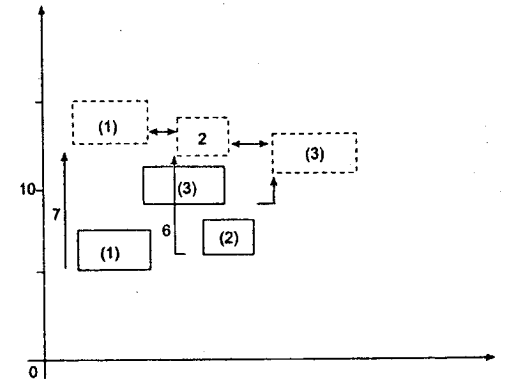
programs (SMOOTH, COPITCH, MODADJUST) transform an object containing sounds of different pitches, glissandos into an object containing only sounds that belong to a desired key: major, naturally minor or melodical minor.[1] The program is useful in *Modal Adjustment* (MA) (2.3.9).

5.2.9. SHIFT

A collection of objects being given, expressed as a matrix of 3rd degree, possibly a family of images generated by the same object subjected to a transformation with different values given to parameters (see in this connection the HOMOTHETY and ROTATION programs), SHIFT translates the collection elements in the *Ox* and *Oy* axes directions.

When the collection has *N* elements, the program requires *N*–1 parting in time between the different elements and *N* partings at pitch level (Fig. 5.3).

Fig. 5.3



The rectangles plotted with solid line on the figure represent the original collection of the image; the rectangles plotted in dotted lines represent the "rearranged" images after transformation. The above result is generated by the program using 2, 3 as time intervals and 7, 6, 2 as translations on pitch scale.

Expressed in dyadic form

F_1 SHIFT F

the program creates the F_1 variable, representing the new "arrangement" in the sound space $\mathbf{R}^2$ of the collection *F*.

[1] Keys are indicated by the notation A, B, C, etc.

5.2.10. SPREAD

Somewhat similar to SHIFT, the SPREAD program allows the arrangement in the specified mode by the user of the *occurrence moments* of a collection elements, i.e., of the X values of points representing these moments. These values are defined by specifying the intervals separating the moments of occurrence (Fig. 5.4).

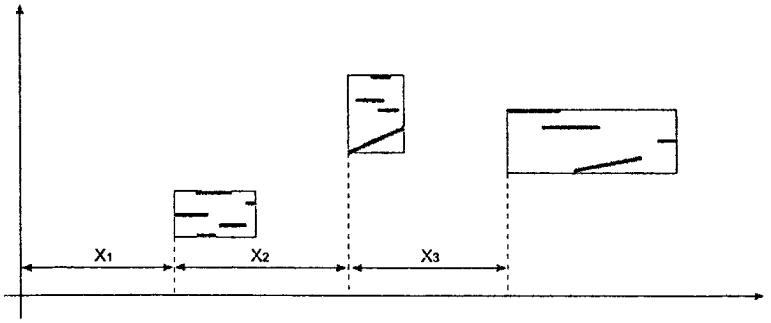


Fig. 5.4

In particular, the program is useful when desiring to set the occurrence moments in a uniform way: $X_1 = X_2 = X_3$. Each of these X intervals should be larger than the corresponding element in order to avoid overlappings of sounds.

Example

Let F be a collection of four objects (forms):

$F_1 \leftarrow$ 0 30 30 30 SPREAD F

The number of these values should be equal to the number of elements of F zero sets the first element to origin.

5.3. PROGRAMS OPERATING WITH COMPOUND TRANSFORMATION

5.3.1. TRAJDATA and TRAJECT

The TRAJDATA program allows the definition of a trajectory with several branches on which the reference point *PR* of an object moves. The trajectory is situated in the virtual sound space $\mathbf{R}^3$ in a parallel plane with the real sound space $\mathbf{R}^2$. The program carries out continuous time transformations of sound forms of the type described under (4.7.1).

During its translation on the trajectory, the object is subject to the following transformations:

1. Homothety	Homothety centre (COT) situated in *PR*; homothety ratio q.
2. Homology	Homology axis (Aho) defined by the point *PR* and angle α made by axis with *Ox* (i.e., base of the rectangle representing the object), homology center (CML) defined by the x_1, y_1 coordinates. Angle ϕ_1 of projections when there is a parallel homology; homology ratio q_1.
3. Rotation in trajectory plane	Rotation centre situated in *PR* and rotation angle β.
4. Rotation in $\mathbf{R}^3$	Rotation axis *Ar* defined by *PR* and angle of axis with *Ox* (i.e., lower side of rectangle representing the object). Rotation angle of object around the axis (Δ).
5. Projection from $\mathbf{R}^3$ in $\mathbf{R}^2$	Projection center (*CP*) defined by the x_2, y_2, z_2 coordinates. Angle ϕ_2 if this is cylindrical projection.

TRAJDATA requires from the user:

1. Trajectory coordinate "knots" ("points") (x, y). For the trajectory beginning point (knob) the value of z coordinates (plane height in $\mathbf{R}^3$) is also required.
2. The value of considered variables of transformation should be specified for each "knot".

At knot 0 (trajectory beginning), by definition:
$q = 1, \alpha = 0$, parallel homology $\varphi_1 = 90$, $Q = 1$,
$\beta = 0, \gamma = 0, \Delta = 0$, cylindrical projection $\varphi_2 = 90$ (Fig. 5.5).

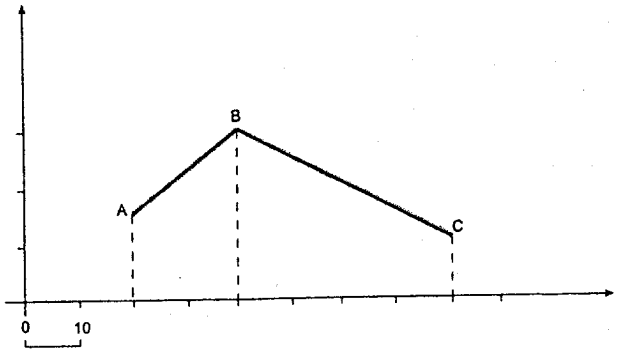


Fig. 5.5

where:

A

$x = 20$ $y = 15$ $z = 20$ $q = 1$ $\alpha = 0$
$\varphi_1 = 90$ $q = 1$ $\beta = 0$
$\gamma = 0$ $\Delta = 0$ $\varphi_2 = 90$

B

$x = 40$ $y = 30$ $q = 1.7$ $\alpha = -10$ $x_1 = 0$ $y_1 = 0$
$Q = 0.7$ $\beta = 30$ $\gamma = 0$ $\Delta = 15$ $x_2 = 50$ $y_2 = 30$
$z_2 = 50$

C

$x = 80$ $y = 10$ $q = 0{,}5$ $\alpha = -30$ $x_1 = 110$
$y_1 = -20$ $Q = 2.25$ $\beta = 45$ $\gamma = -5$ $\Delta = 30$
$x_2 = 90$ $y_2 = 0$ $z_2 = -10$

N.B. The angle values at different knots are effective values, and not increases or decreases relatively to values from the preceding point.

Interactively, the TRAJDATA program requires from the user the values of the x, y coordinates of each knot (with $knot_0$ the z value, height of trajectory plane, should be inserted), as well as whether the user desires or not to get further with another branch.

Here is the mode of operation for our example:

	x	y	z	?	
$knots_0$	20	15	20		
$knots_1$	40	30	–	?	z is indicated with $knot_0$ only (position of plane P relative to sound space $\mathbf{R}^2$)

OMOL ratio (q), axis angle OMOL (α), OMOL center x_1, y_1[2] or projections angle with axis (ϕ_1), OMOL ratio (q), rotation angle in $\mathbf{R}^2$ (β), rotation axis angle with Ox (γ), axis object rotation angle (Δ), projection center (x_2, y_2, z_2)[3] or projecting angle with $\mathbf{R}^2$ (ϕ_2)?

1,7 –10 – –90 0.7 30 0 15 50 30 50 –
Another branch?

[2] When OMOL center is situated at finite distance, the centre coordinate values x_1, y_1 are inserted. With parallel projections φ_1, projection angle with homology axis only is given.

[3] In case projection center CP is at finite distance x_2, y_2, z_2 are given. With cylindrical projection, angle φ_2 with sound space $\mathbf{R}^2$ is given.

YES
Point$_2$ x y z
80, 10

OMOTratio (k), OMOL axis angle (α), OMOL center x_1 y_1, or projection angle with axis (ϕ_1), OMOL ratio (q), rotation angle in $\mathbf{R}^2$ (β), rotation axis angle with Ox (γ), axis object rotation angle (Δ), projection center (x_2, y_2, z_2) or projection angle with $\mathbf{R}^2$ (ϕ_2)?

0.5 –30 110 –20 –2.5 45 –5 30 90 0 –10
Another branch?
NO
Matrix name?
T
(The program generates a variable as a matrix form.)
For our example, this matrix has the following form

T	x	y	z	k	α	x_1	y_1	ϕ_1	q	β	γ	Δ	x_2	y_2	z_2	ϕ_2
$Knot_0$	20	10	30	1	0	–	–	90	1	0	0	0	–	–	–	90
$Knot_1$	40	30	–	1.7	–10	–	–	90	0.7	30	0	15	50	30	50	–
$Knot_2$	80	10	–	0.5	–30	110	–20	0	2.5	45	–5	30	90	0	–10	–

Any number of such trajectories may be saved under various names. A trajectory may have any number of branches.

TRAJECT is the program used to make the translation of an object on a trajectory. The program requires the reference point of the object which may be defined no matter where in object rectangle. Further on, it needs a value x (included in the minimum and maximum x values of the trajectory). The program selects the branch with the indicated x value. The variable values on each branch change from the values specified with the origin point of the respective branch to those specified with the next point, proportionally to the distance covered on trajectory in the Ox axis sense.

Numerical example

Let 55 be the value x (moment of object mapping in its motion on trajectory). Since the abscissa 55 is on branch 2, defined on Ox axis by the interval (40, 80), the homothety ratio k value (which, during the considered interval diminishes from 1.7 to 0.5 (1.2 units) will have at the $x = 55$ moment the value

$$k = 1.7 - \frac{15 \times 1.2}{40} = 1.25$$

The values of other variables are calculated in a similar way.

5.3.2. MOTION

The MOTION program allows the carrying out of some continuous transformations depending on the speed at which a sound object moves during the chronological time unit and the moments when the object images are picked up. The program operates in the virtual space $\mathbf{R}^3$ on an object moving on a trajectory defined as we will see by the object direction and speed of motion in the sense of Ox axis or Oy axis, respectively, and which rotates in plane around the reference point and in space around an axis defined by the reference point and angle with Ox. The object rotation speeds in plane, of axis around the reference point and of object around the axis are also expressed "picked up" on chronological time. The moments where the object is "picked up" in motion may have arbitrary values.

The program requires from user three "sets" of information:

I

Initial value[4] of coordinate x
Speed in Ox sense
Initial value of coordinate y
Speed in Oy sense
Initial value of rotation angle in plane
Rotation angle speed
Initial value of axis angle with Ox
Axis angle speed
Initial value of rotation angle around axis
Rotation angle speed around axis.

II

The moments (t) when the image of the object in motion has to be mapped.
For instance: 3 5 13 17

III

The x, y, z coordinate values defining the object reference point in $\mathbf{R}^3$.
The coordinates of projection center or, with cylindrical projection the projection point situated to infinity, or the projections angle with $\mathbf{R}^2$.

[4] "Initial value" means the value of the respective variable m at moment $t = 0$.

5.4. PROGRAM TRANSLATING IN SCORE SYMBOLS MUSICAL SOUND FORMS REPRESENTED NUMERICALLY IN SOUND SPACE R^2

5.4.1. PART

The program achieves the transposition of sound forms digitally represented by matrices (2.1.3) in musical notation.

By means of the generator of special characters symbols specific for musical writing were obtained: note values, accidents, pauses, the "legato" sign, as well as four types of arrows.

The user specifies the matrix defining the object, bar value and desired key for writing the score.

The program transforms the bar into a number of semiquavers, stored by the TACT variable. The matrix is arranged according to the x values (i.e., according to time: within a matrix line, when the occurrence moment is smaller than the disappearance moment of the sound ($x_1 - x_2$), then, SENS = "——>", otherwise, SENS = "<—— " (required in case the sound changes its sense through transformation, for instance, a rotation (2.5.1)).

The program traverses the matrix. For each sound, the number of the beginning bar, of the ends in the respective bar or if it continues in the following bar (or bars), the character (type) of the note representing the duration, possible connecting durations, sound pitch, are determined, as well as the glissando and the point where the glissando is carried out.

The matrix line defining the sound whose occurrence moment is the nearest to the disappearance moment of the relevant sound, is determined. Thus, it is also determined whether the considered sounds are superposed or not and, if so, if they should be written on separate staves. The succession numbers of the sounds related to the initial matrix lines are also specified.

The following functions are used: INT – the whole part of a number; INP – the nearest lower whole; ABS – the absolute value of a number.

The program also makes use of four subroutines:

PROX – determining the deviation from the beginning or end of a sound represented by a certain symbol of musical duration. (As a rule, the sounds resulting from various transformations, have equal durations to those obtained by the division of a fundamental duration. Such being the case, the "note" symbol does not mean the sound effective length of time (duration). PROX amends this possible inadequacy between the symbol-sign and the significant sound.)

DIF = X – INT(X)
When ABS(DIF) 0.17, then T = " " (blank)
when DIF = 0, then T = " "
otherwise
otherwise T = " "

BINAR – determines the number of whole notes, minimums, crotchets, quavers and semiquavers (expressed as powers of 2) "entering" in a given time interval. The subroutine determines the Point values and the number of points.

NOTE – establishing the note pitch
DY = Y – INT(Y)
When DY 0.17, then Y = INT(Y)
SJ = " " (blank)
thus when 1 – DY 0.17, then when DY 0.5, then y = INT(Y + 1)
SJ = " "
otherwise Y = INT(Y)
SJ = " "
otherwise Y = INT(Y + 1)
SJ = " " (blank)

Thus, this "new" Y represents the number of semitones related to the origin of the pitch axis D_{03}.

The variable DY shows the deviation related to the pitch of the nearest "temperate" sound. On the basis of Y: the octave where the sound "falls", the note denomination, as well as the possible sharp or flat are determined.

NMS – increases the bar number (index) by one unit and, when the result is zero, it transforms it into 1. This is necessary in order to avoid a bar with index 0.

6. MUSIC AND POLYMODULAR TIME (STRUCTURAL POLYTEMPI MUSIC)

6.1. GENERAL REMARKS AND SOME CONSIDERATIONS REGARDING MUSICAL TIME

This part deals with a specific domain of music, namely musical time, more precisely with musical structures in which time is measured simultaneously with different time units. In more usual musical terms, we would say that it deals with music in which musicians or singers play or sing simultaneously according to different "beats", each of these "beats" being endowed with a different tempo. By beats musicians understand a row of periodical accents occurring in time, expressed by rhythmically uttering the sound symbols of a row of numbers, or, for large ensembles, by the gestures of the conductor. The speed at which these beats unfold in time represents the "tempo" expressed as a number of such beats per minute. Making use of a terminology introduced in some papers that I have published previously, we would say that this part is concerned with Music in Polymodular Time or Structural Polytempi Music.

Everybody knows that the duration of sounds in music is measured with respect to a certain time interval which occurs between two consecutive beats and which is consedered as "unit of measurement."

When dealing with traditional European music, one finds that in music played by an ensemble, all members of the ensemble measure the time of the music of their parts by the same time unit, i.e., with respect to the same beats. In usual musical terms, we would say that at a given moment, all members of that ensemble play according to the same fundamental beat, hence, in the same general tempo.

However, we must emphasize that we deal with a particular case of music, the traditional European music, and that the music of some other cultures, like that of ancient China, as well as the folk music of certain nations, do not display this oneness of time measuring unit common to all members of a playing group. This somewhat poor rhythmical structure of western music is partly due to the restriction given by a staff notation which does not allow a notation in a very easy and precise manner of music in which the different "voices" (parts) unfold at different tempi.

Towards the end of the 19th century, certain comporsers started to be concerned with the idea of replacing, on certain intervals, the oneness of the beat of an ensemble by the superposition of layers endowed with beats having different tempi: Gustav Mahler, Richard Strauss, Charles Ives.

Later on, the problem was resumed by other contemporary composers. As a rule, the attempts to compose such music were not directed towards obtaining some rigorous structures, the results tending to more or less random arrangements. However, when well-defined structures were aimed at, they did not go beyond some special cases characterized by a correlation of the tempi for different structures, hence, implying their mutual dependence. An exemple is the correlation of some tempi whose metronomic values admit a common divisor, a common "beat" being thus periodically obtained for the various layers. In most of these attempts the composers used such procedures in one or more of their works, merely as a curiosity, with no greater implications in their compositions process. With very few exceptions, they did not write entire pieces in such a mode; in most cases, they confined themselves to intervals defined by some "beats", some bars or some passages. Even in these cases, the proper performance during a concert of such scores was not possible, since there existed no adequate technology and the movements of several conductors could not be satisfactorily co-ordinated.

The limited ways in which composing music in Polytempi seemed possible, and the impossibility of performing such works in a rigorous mode have, in the main, discouraged composers to enter upon this domain, and made those who had approached it abandon it after a few attempts.

There was, on the one hand, the lack of a general theory providing the conception of some well-defined musical time structures requiring layers with different tempi values – constant or variable – and, on the other hand, the lack of a technology providing a rigorous performance during the concert of musics written in several simultaneous tempi.

The Polymodular Time theory and the associated technology, the Polytimer theory and technology to be presented in this part, are meant to solve both the fundamental, theoretical aspect related to writing music in Polytemporal structures (Structural polytempi), and the practical aspect of actually performing this type of music in concerts.

6.2. POLYMODULAR TIME

6.2.1. Basic ideas concerning the polymodular time

Defined as a general model of time structures, the Polymodular time is characterized by the property that time is simultaneously measured with

respect to various time intervals, considered as "units of measurement" and referred to as "time modules".

The mathematical model of the polymodular time allows the study of some complex time structures existing in nature, proper to astronomical, physical, biological, social and technological phenomena, as well as in the domain of arts, of music, choreography and theatre, of the static and kinetic visual arts and of architecture.

In particular, the Polymodular time concept furnishes a solution to the problem of generating polytempi musical structures resorting to specific methods.

The polymodular time theory was first brought out by the author at Syracuse N.Y. University in 1970, and subsequently published and reviewed in various journals and news media. The technical device required to apply polymodular time in art (structural polytempi), the Polytimer, was patented by the author in 1970. Demonstration performances, including music and choreography based on these principles, were presented at Syracuse (1970), Bucharest (1977) and at the international festival "Printemps Musical de Paris" (1978).

6.2.2. Introductory concepts

Time – a fundamental dimension of life and of universe – has been studied in the course of the history of science and philosophy in regard to its irreversibility and duration values.

The search for a rigorous and complete definition of time would always be a concern of mankind. Any attempt at describing "pure time", relating it to other concepts like space and motion is usually equivocal, on account of the ambiguity of the term "duration". Different "clocks" correspond to different "events" of respective character, hence different time reference systems imply different time mesuring units. The "year" of different planets has different "durations" according to their own time of revolution. On a given planet, a day's duration changes day by day because of the differences in sunrise and sunset. On a certain date, the day's duration has different values at different latitudes. The duration of life of various animals and plants is measured in different ways by their own "internal clocks". As a result, the duration of various biological events in different animals, as compared to their life span, has different values.

The abstract concept of duration itself has different meanings and interpretations in mathematics and philosophy.

In the scientific sense (according to Einstein), an objective duration is variable in terms of the relative motion of the object related to an observer. According to Bergson's psychological theory of time's apparent flow, duration is subjective and the value of a certain time module is also variable as related to certain "states", as are, for example, the age of a person or that person's psychological condition.

In all the above-mentioned cases, for a certain event or for a certain person, at a givel moment, in a given place and in a given physical or psychic condition, time flow could be measured by a certain unit of measurement, by a specific "time module".

Consequently, time, related to specified events or persons, appears to a certain observer as having a *monomodular* structure, in the sense that, at that given moment, the measurement of its flow is perceived in terms of a *unique, specific time module.*

However, for an imaginary observer who could simultaneously experience, at a given moment, the various modules by which different events or persons, in different states, define time intervals, the time would have a *polymodular* structure. This means that, at a given moment, the time flow would be *simultaneously measured by means of a finite set of modules.* Then, time structure is defined for the observer by the totality of the relations among the elements of the corresponding set. These relations may appear as a Cartesian product of finite intervals, a product that can be viewed as an *n*-dimensional parallelepiped, which represents, at a given moment, the time polymodular structure, as related to a certain number of phenomena or persons. In a given interval the size of this *n*-dimensional parallelepiped may change continually, since the value of each module may vary in terms of the involved objective or subjective condition. The variation of the module value being frequently a function of random factors, the size of the *n*-dimensional parallelepiped appears as having a probabilistic character. This fact emphasizes the time statistic aspect, which is detected at the chronological time level, where the periodicity of the phenomena leading to its measurement frequently shows random, though infinitesimal, variations.

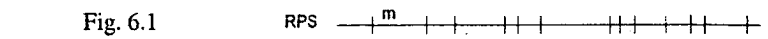
6.2.3. Definitions

The time module (*m*) is defined as the time interval between two consecutive real pulsations.

By "Real Pulsation" (*RP*) we mean the moment of emergence of any phenomenon ("event") with periodical or non-periodical repetitive

character.[1] A finite and strictly ordered set of such pulsations unfolding in time represents a "Real Pulsation System", *RPS* (Fig. 6.1).

Fig. 6.1



The mode of distribution of a number (*n*) of pulsations of an *RPS* in a given time interval (*I*) represents the Law of Distribution (*L*) of the *RPS* over *I*. In music, this distribution law is defined as tempo (*Tp*).

A finite set of *RPS* forms an *RPS* collection (*C*). *C* may have a single element or be empty[2] (Fig. 6.2).

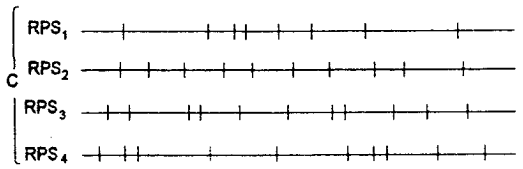


Fig. 6.2

A Polymodular Time Structure, *PMT*, is defined as a collection of real pulsation systems having the following properties.

1. *C* is endowed with a Virtual Pulsation System, *VPS*. The *VPS* is a reference system for *C*: Time Reference System *TRS*.

The (*M*) time module is defined as the time interval between two consecutive virtual pulasations. As a consequence, the various time intervals *I* in *RPS*, and *L* specifying the *RP* distribution on them, are defined with respect to the pulsations of the *TRS* (in general, with respect to pairs of such pulsations, whether consecutive or not).

The virtual pulsations from *TRS* are provided with a virtual law of distribution (*VL*). If the pulsations of the *TRS* are periodic, the *TRS* is a time measuring system for *C* and the interval between two (*M*) successive pulsations becomes a "meter", a time-measuring unit for *C* (Fig. 6.3).

2. There is a set of *real* pulsations, specified in *C*, that are synchronous with certain *virtual* pulsations, specified in the *TRS* (Synchronous

[1] In this respect, the waves rhythmically breaking against the coast, the blows of a power hammer, the revolution motion of some objects or of the planets, the lighthour signals, the heart, the human pace or certain rhythmical movements are considered to be "events". As far as music is concerned, the in-time occurrence of a musical sound is considered an "event".

[2] In the given examples, *x* represents the virtual pulsation; →← denotes the real individual pulsation.

Pulsations, *SP*).[3] The *SP* set induces in *C* a general pulsation system, referred to as Fundamental Pulsation System, *FPS*.

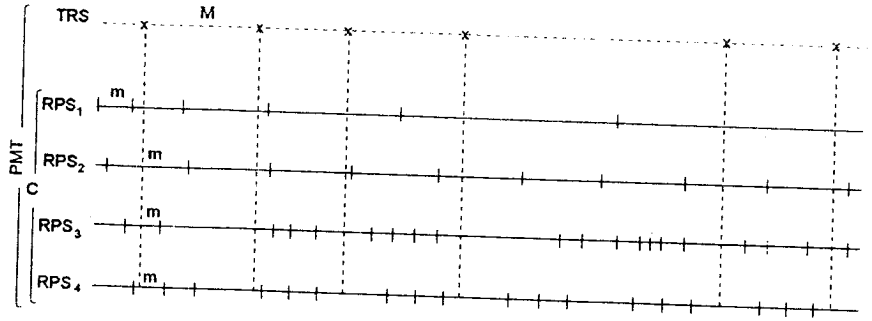


Fig. 6.3

The value of the cardinal of the *FPS* set *defines the degree* of coherence of the *PMT*.

The cardinal of the set of real pulsations in *C* synchronous with a *specified virtual* pulsation in *TRS* defines the degree of *local coherence* of the polymodular time structure at the moment defined by the given pulsation in *TRS*.

Two or several *RPS* belonging to a given collection may exhibit synchronous pulsations, occurring either by chance, or because of the existence of some implicit relations between their structures.[4] We shall refer to such pulsations as Spontaneous Synchronized Pulsations, *SSP*. The set of spontaneous synchronized pulsations *SSP* can induce in *C* a Spontaneous Fundamental Pulsation System, *SFPS*, and the value of its cardinal can define different *degrees of spotaneous coherence*.

6.2.4. Degree of complexity

By definition, the *PMT* is considered to have *degree of complexity I*.

A *PMT* collection provided at the respective *TRS* level with a general system of virtual pulsations, the Time Reference Hypersystem, $[TRS]^2$ is a

[3] In practice, depending on the repetitive events whose emergence moments are considered as pulsations, two pulsations may be considered as synchronous when the time interval between their emrgence points takes a very small value ε.

[4] In the latter sense, for instance, when two systems have isochronous pulsations with different tempi over the same interval *I*, and the corresponding numbers of pulsations over *I* of the two systems admit a common divisor, the systems include one or several synchronous pulsations.

polymodular time structure of *second degree* of complexity, $[PMY]^2$. $[TRS]^2$ is a "time reference system of the time reference systems" (Fig. 6.4).

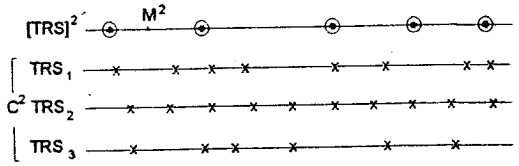


Fig. 6.4

The interval between two $[TRS]^2$ consecutive pulsations represents the M_2 *module*. The distribution law of the $[TRS]^2$ pulsations represents the hyperlaw $(L)^2$ (Hypertempo *HP*).

A collection of $[PMT]^2$ provided, at the level of their respective $[TRS]^2$, with a general, common virtual pulsation system, $[TRS]^3$, forms a polymodular time structures of *degree of complexity 3*, $[PMT]^3$.

By hierarchically organising collections of $[PMT]^n$, by providing the respective $[TRS]^{n-1}$ with a common time reference system $[TRS]^n$, a polymodular time structure of successively higher degree of complexity can be built up.

A complex polymodular time structure $[PMT]^n$ may be represented as an *oriented graph*, each node representing a certain real pulsation system (time reference system) of a specified degree $[TRS]^i$ (i = 1, 2, ..., n) (see Fig. 6.5).

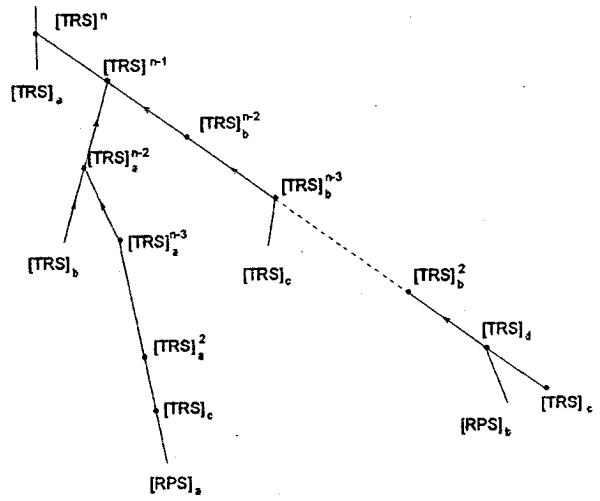


Fig. 6.5

The *degree* of a *TRS* is defined by the number of edges in the longer line connecting one of the *RPS* to the respective $[TRS]^i$. The *TRS* of

maximum degree ($[TRS]^n$) *defines the degree of complexity* of the given polymodular time structure.

The *degree of dependence* may be also defined for each $\{RPS, [TRS]^i\}$ pulsation system, expressed by the number of edges connecting a given pulsation system $\{RPS, [TRS]^i\}$ and the *TRS* of maximum degree $[TRS]^n$. For instance, in Fig. 6.5, TRS_a has dependence degree 1, since its time reference system is $[TRS]^n$ itself; TRS_b has dependence degree 3, since the intrinsic time organisation of its pulsations may be related to the $[TRS]^n$ by passing only through $[TRS]_a^{n-2}$ and $[TRS]_a^{n-1}$; TRS_c and TRS_d have degree of dependence n–1; RPS_a, RPS_b and RPS_c have degree of dependence n.

The collections of $[TRS]^i$ (i = 2, 3, ... ,n) may have various *degrees of coherence*, according to the value of the cardinal of the set of virtual pulsations specified in $[TRS]_i$ synchronous with certain virtual pulsations specified in $[TRS]^{i+1}$.

A Fundamental Pulsation System of degree i $[FPS]^i$, of the various time reference systems, $[TPS]^i$ involved, is induced by these pulsations in the respective collections of virtual pulsation systems.

The degrees of coherence may occur at the level of a $[PMT]^n$ structure between the various collections of virtual pulsation systems $[TRS]^{n-1}$ and the respective $[TRS]^i$ systems dominating them.

6.2.5. Some basic degrees of coherence of polymodular time

The degree of coherence of the polymodular time is the power of the relation established between a finite number of pulsation systems and the reference system dominating them through the agency of the fundamental pulsation system, *FPS*. Since, at the level of *PMT* structures of degree 1, the real pulsations represent the occurrence time of some phenomena-events, the degree of coherence shows the power of connection existing at the time-parameter level between a finite number of events simultaneously displayed.

In what follows, we shall consider a *PMT* structure as having degree of complexity 1.

Depending on the relations that can be established between the pulsations of the time reference system, *TRS*, and the real pulsations of the collection *C*, that is the mode in which the Fundamental Pulsation System (*FPS*) is defined, a polymodular time structure (*PMT*) may have the following four basic degrees of coherence.[5]

[5] In the following examples, ↑ represents the fundamental real pulsation; ↓↑ denotes spontaneously synchronized pulsations. *I* stands for the pulsation distribution law over an interval *I*, *nP/m* is the number of pulsations per minute.

I. The *FPS* set is empty; none of the *C* pulsations is synchronous with a pulsation in *TRS*. We shall say that the *C* collection has a Totally Virtual Fundamental Pulsation System, *TVF*, and that the *PMT* has degree of coherence I (see Fig. 6.3).

II. A proper part of the *TRS* elements is synchronous with certain pulsations belonging to a proper part of the *RPS* elements in *C*. Such being the case, we may say that *C* presents a Partially Virtual Fundamental Pulsation System, *PVF*, and that *PMT* has degree of coherence II (see Fig. 6.6).

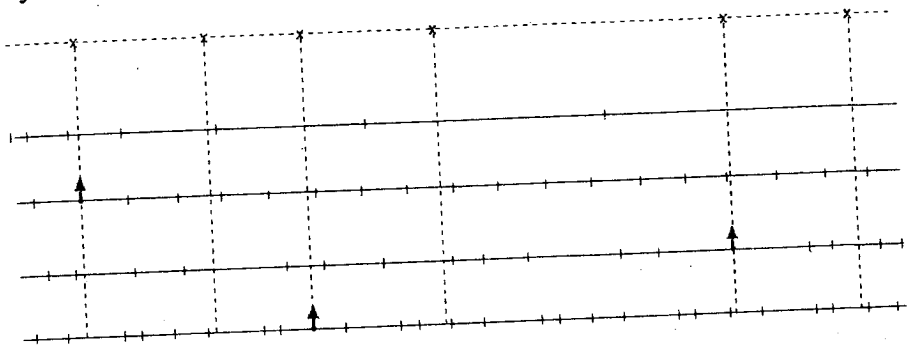


Fig. 6.6

III. Each element in *TRS* is synchronous with a pulsation defined in a proper part of the *PS* elements in *C*. Then *C* is said to present a Partially Real Fundamental Pulsation System, *PRF* under consideration has degree of coherence III (see Fig. 6.7).

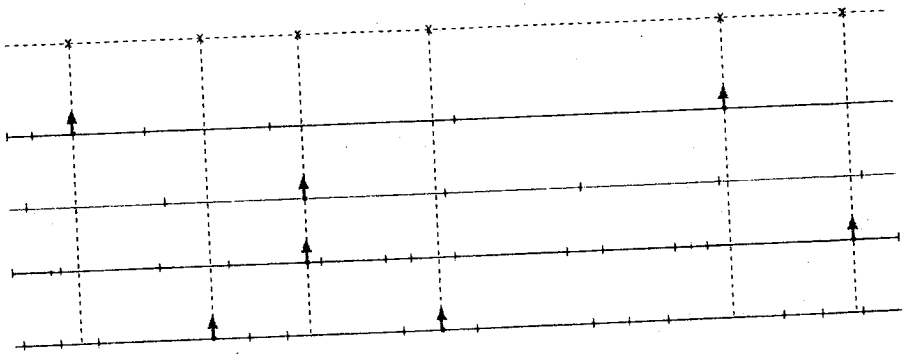


Fig. 6.7

IV. Each element in *TRS* is synchronous with a pulsation defined in each *RPS* element in *C*. In this case, we may say that *C* presents a Totally Real Fundamental Pulsation System, *TRF*, and that *PMT* has degree of coherence IV (see Fig. 6.8).

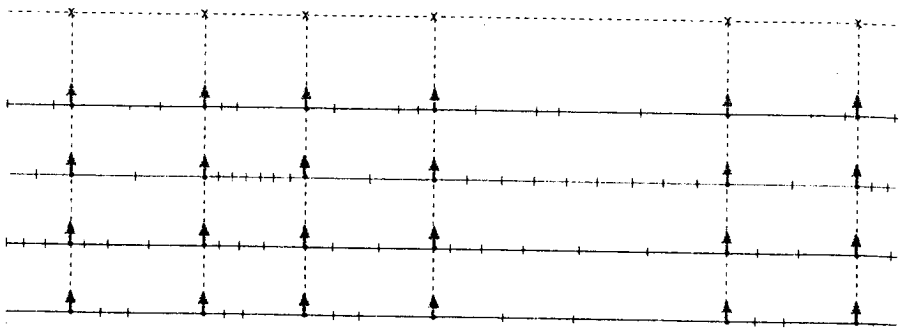


Fig. 6.8

Many times structures in the physical and biological world, as well as many manifestations of life can be interpreted as having a Totally Real Fundamental Pulsation System organisation.

Although life is a continuous phenomenon, its biological, physical and social manifestations are discrete. As was already noted, at an individual level, the various types of rhythms, such as heart beats, walking and various rhythmical movements during work, are regarded as "pulsations". At the social-life level, for instance, for the workmen in an enterprise, the beginning and end of their activity at specified hours, for a certain public, the beginning and end of performance, for a certain group of travellers, the departure and arrival of the train, can be viewed as collective "pulsations", whose laws of distribution are defined by "schedules" of the respective actions. It is easy to notice that, on the one hand, a certain person may be simultaneously implied in several temporal systems, and, on the other hand, that in a finite set of such systems, a polytemporal structure may be determined, having sometimes a Totally Real Fundamental Pulsation System.

A special case of Polymodular Time structure endowed with a Totally Real Fundamental Pulsation System is the one described below.

If in a polymodular time structure (1) the collection *C* is provided with a Totally Real Fundamental Pulsation System, (2) the tempo of the pulsations of the *TRS* has a constant value (isochronous *M*) and (3) each *RPS* develops between two consecutive pulsations of the fundamental system a constant number of identically organized individual pulsations, then the structure presents a *recurrent* or *cyclic* polymodular organisation.

This type of polymodular time structure may be detected in many natural phenomena. For instance, the Greek astronomer Menton (Vth century B.C.) found that an integer number of lunations (lunar periods), i.e. 235, is included in a period of 19 years. Comparing the revolution phenomenon with the concept of "pulsation", Fig. 6.9 represents the

"Menton cycle", where the 3 *RPS* represent a revolution of the earth around the sun within a 19-year interval (1), of the moon around the earth (2) and of the earth around its axis (3).

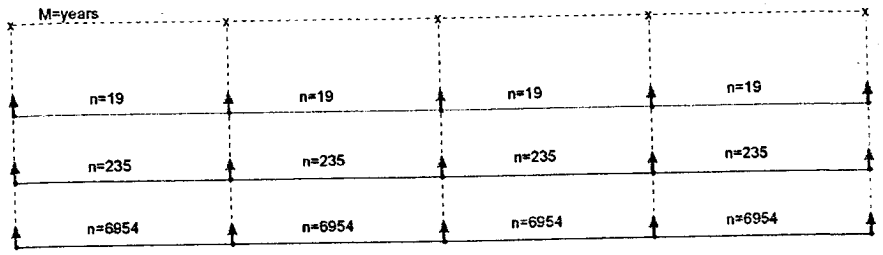


Fig. 6.9

Similar types of temporal organisation may be found in the dynamics of many other structures too.

Time organisation in a cyclical Polytempi seems to be essential in the thought of ancient peoples. In the cosmic-mythological concepts, the time was provided with a periodical "regeneration" capacity, which is inconsistent with the modern irreversible-time concept. This ontological aspect of the time is caught by Mircea Eliade[6]: "It is essential for us to realize that there is everywhere a concept of an end and of a beginning of *temporal periods*, based on the observation of some biocosmical rhythms." "... the existence of a periodical creation concept, namely of a cyclical regeneration of the time."

The end and the beginning of those periods are marked by ceremonies and rituals, whose aspects vary with era, place or society, but all being based on the concept described by Mircea Eliade as "the cyclical return of what has been before, in one word, of the 'eternal return'." Further on, Mircea Eliade observes the existence of the "repetition theme in an over-typical gesture cast on all the cosmic, biological, historical, human, and like planes," but, at the same time, the "time cyclical structure regenerating with each new 'birth' on any of these planes" too.

"The past is the premonition of the future."

Obviously, a practically infinite number of cosmic, biological, physical and social pulsations simultaneously develops between the pulsations marking the great periods, the great temporal cycles. The cycle "pulsation" may be interpreted as a "fundamental pulsation" which, in a broad sense, defines a recurrent polymodular time structure.

[6] Eliade M., *Le mythe de l'éternel retour*, Gallimard, Paris, 1969.

6.2.6. Spontaneous fundamental pulsations systems

A set of spontaneously synchronized pulsations may induce a Spontaneous Fundamental Pulsation System, *SFPS*. Depending on the *SSP* set cardinal value, we shall say that *PMT* has a totally virtual (I), partially virtual (II), partially real (III) or totally real (IV) spontaneous degree of coherence (see Fig. 6.10 – *Totally Real, SFPS*).

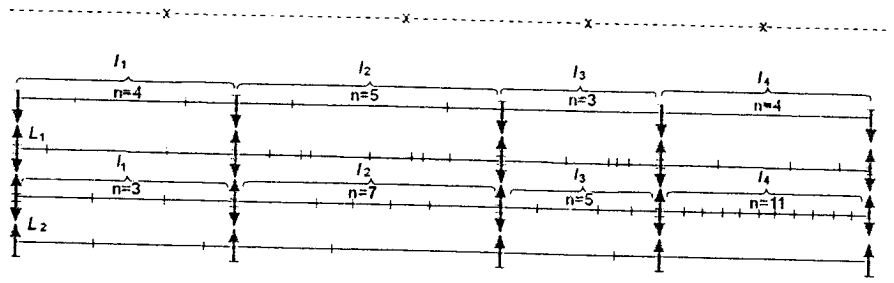


Fig. 6.10

6.2.7. Degrees of coherence at the level of the time reference systems in $[PMT]^n$, $n > 1$

The same special cases of organisation may occur between the different $[TRS]^{i-1}$ collections and the respective $[TRS]^i$ system dominating them. The following examples will illustrate this.

6.2.8. Examples of polymodular time structures with degree of complexity n ($n > 1$)

Fig. 6.11 presents a polymodular time structure of degree of complexity 2. The collections of Real Pulsation Systems (*RPS*), C_a and C_b, are provided with the Time Reference Systems TRS_a and TRS_b, respectively, thus forming two polymodular time (*PMT*) structures, A_a and A_b, respectively. The TRS_a, TRS_b collection is endowed with a time reference hypersystem $[TRS]^2$. $[TRS]^2$ is a general time reference system for the A_a, A_b collection. The pair (A_a, A_b), $[TRS]^2$ is a polymodular time structure with degree of complexity 2 $[PMT]^2$, A^2 (see Fig. 6.11).

As far as the degree of coherence is concerned, A^2 has, at its various levels of the time reference systems, the following characteristics:

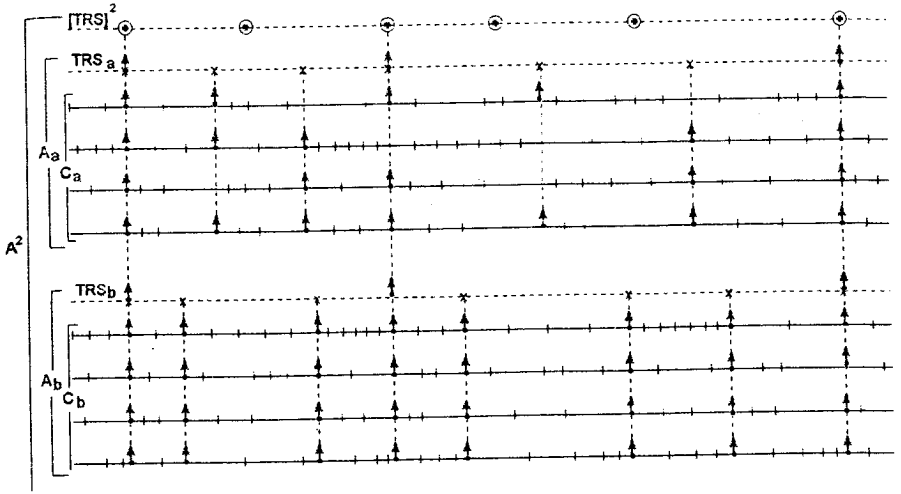


Fig. 6.11

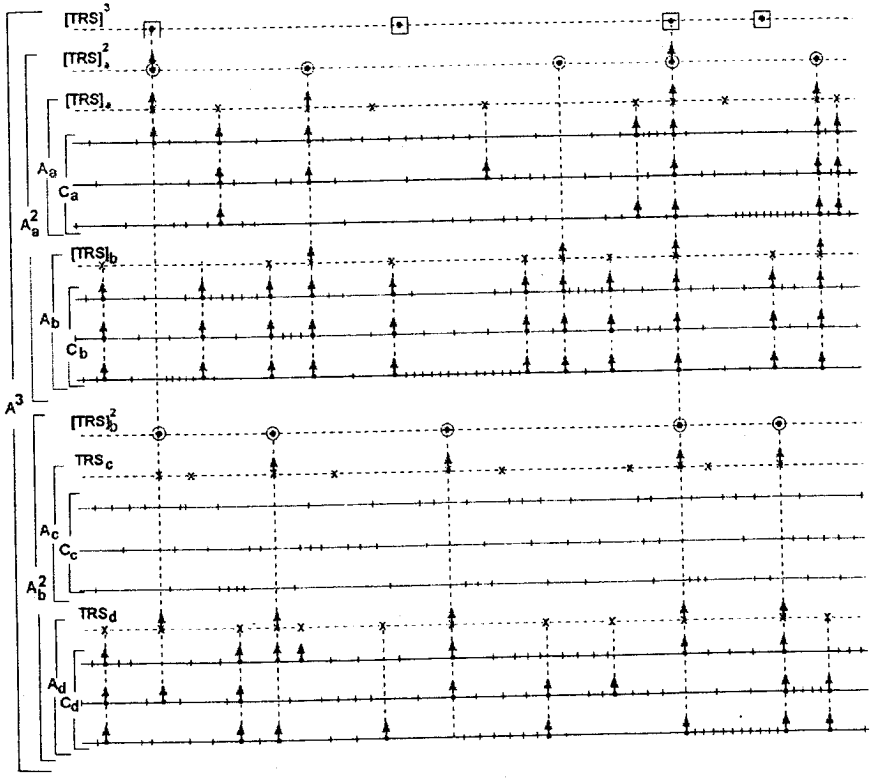


Fig. 6.12

(TRS_a, TRS_b), $[TRS]^2$ – Partially Virtual Fundamental Pulsation System of second degree, $[PVF)^2$, defined by the set of virtual pulsations specified in the TRS_a, TRS_b collection synchronous with certain virtual pulsations specified in $[TRS]^2$.

C_a, TRS_a – Partially Real Fundamental Pulsation System, *PRF*, defined by the set of real pulsations specified in the C_a collection, synchronous with certain virtual pulsations specified in *TRS*.

C_b, TRS_b – Totally Real Fundamental Pulsation System, *TRF*, defined by the set of real pulsations specified in C_b, synchronous with certain pulsations specified in TRS_b.

Fig. 6.12 illustrates polymodular time structure with degree of complexity 3 $[PMT]^3$.

Four collections of Real Pulsation Systems C_j (j = a, b, c, d) are provided with the time reference systems TRS_j, thus forming the polymodular time structures A_j. The collection A_a, A_b, *at the level of their reference systems* TRS_a, TRS_b, is provided with a general time reference system $[TRS]^2_a$, the collection A_c, A_d with a general time reference $[TRS]^2_b$. The pairs $((A_a, A_b), [TRS]^2_a)$ and $((A_c, A_d), [TRS]^2_b)$ form $[PMT]^2$ A^2_a and A^2_b. The collection A^2_a A^2_b at the level of their reference systems $[TRS]^2_a$ and $[TRS]^2_b$ is provided with a general time reference system $[TRS]^3$. The pair $((A^2_a, A^2_b), [TRS]^3)$ forms a polymodular time structure with degree of complexity 3 $[PMT]^3$, A^3.

As far as the degree of coherence is concerned, A^3 has the following characteristics at the level of the time reference system:

$([TRS]^2_a, [TRS]^2_b), [TRS]^3 \rightarrow PVF$

$(TRS_a, TRS_b), [TRS]^2_a \rightarrow PRF$

$(TRS_c, TRS_d), [TRS]^2_b \rightarrow TRF$

Establishment of the degrees of coherence in the structures of degree 1, C_a, C_b, C_c, C_d is left to the reader.

6.3. LAWS OF DISTRIBUTION OF VIRTUAL AND REAL PULSATIONS IN A POLYMODULAR TIME STRUCTURE (PMT)

6.3.1. Case of structures with degree of complexity 1

A. At the level of Time Reference System, *TRS*, the laws of distribution (*L*) of the virtual pulsation (*VP*) are defined over different time

intervals, referred to as segments [*S*]. A segment is represented by an integer number of time modules *M*.

These laws state the values of the various modules *M*, defining the time interval between two successive *VP* of the *TRS*. As a rule, *L* may be expressed as a row of *M*.

If the *M* modules are isochronous (the *TRS* tempo on *S* is constant), then the row of *M* may be expressed:

a) in a *multiplicaive* form (*x*) as

$n \times M$ $(n \in N)$, $\quad (S = n \times M)$

b) by *dividing* a time interval *I* by *n*,

$M = I / n \quad (I = S)$.

In this latter case, the tempo value is expressed by the number of pulsations developing on $I(nP/I)$. When $I = 60'$ (1 minute), the tempo value is expressed in the classical *nP*/min. form. When $I \neq 60$ sec., the tempo is given, in relation to the minute, by 60 sec. $\times n$.

B. At the level of Real Pulsation Systems, *RPS*, the same laws of distribution may be applied, but related to the *TRS*, as, for instance:

a) the emergence point of a given *RP* pulsation (which, represents again, for instance, the first element of a row of pulsations defining, in its turn, a row of time modules m_1, m_2, ..., m_n) is expressed in relation to a virtual pulsation VP_i (i = 1, 2, ..., n) in *TRS*, as

$t = VP_i + x$,

where *x* may be a positive or negative number, or zero;

b) a time interval *I* on *RPS* may be defined:

1) as $RP_I + x$, where RP_i is defined according to a);

2) in relation to two pairs of successive or non-successive *VP*, as

$(VP_i + x), (VP_j + y)$

where *x* and *y* are positive or negative numbers, or zero (Fig. 6.13).

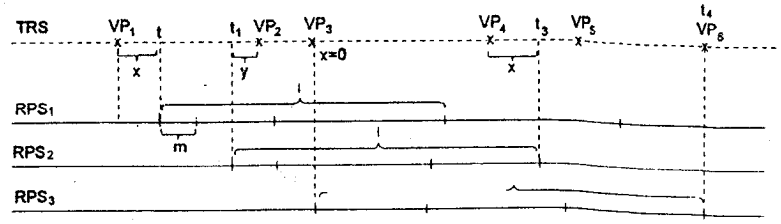


Fig. 6.13

6.3.2. Tempo transformation

A row of m_b, (i = 1, 2,..., n) time modules defining a transition from a row of pulsations with constant tempo (defined by a given m_a module) to another row of pulsations with constant but different tempo (defined by another given m_c module) may be represented in a bi-dimensional space with Ox, Oy coordinate axes, where $x = t$ (time), and $y = r$ (rank of the modules) (see Fig. 6.14).

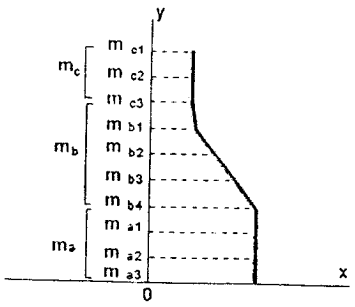


Fig. 6.14

The different ways in which such a transformation of tempo may be carried out can be expressed in terms of a practically infinite number of functions defined by the different intermediate rows of time modules. In a first approach, one would be tempted to make use of arithmetical or geometrical progressions as laws of distribution for such a transformation. But, in a polymodular time structure, one can be confronted with the constraint that the transformation should be carried out in a given time interval. This is raising, on the one hand, relatively complex problems, although, at first sight, the task would seem to be easy; we shall not discuss this aspect here. On the other hand, the manner in which a tempo transition is obtained depends on different variables, which usually induce no arithmetical or geometrical progressions. For instance, a car driver changes the speed of his car (the speed = the tempo of the wheels "pulsations") according to his psychological condition or with the involved road circumstances; a runner – according to the pursued aim or his fatigue condition, etc.

It is therefore obvious that it is essential to determine the number of intermediate time modules n for obtaining a "smooth" transition.

Let us consider three time intervals, I_1, I_2, and I_3, defined on a *RPS*. Let m_a and m_c be two different time modules defining I_a and I_c pulsations, respectively, with a certain tempo each. Let us assume that the tempo on I_a has a lower value than that on I_c. Consequently,

$$m_a > m_c. \tag{1}$$

Each time module m_{bi} (i = 1, 2, ..., n) of decreasing value should observe the following condition:

$$m_a > m_{bi} > m_c. \tag{2}$$

The intermediate row of m_{bi} modules should observe the condition

$$\sum_{i=1}^{n} m_{bi} = I_2. \tag{3}$$

Since a mean value of m_b may be written as $m_b = I_2 / n$ ($n \in N$), (2) can be expressed as

$$m_a > I_2 / n > m_c. \tag{4}$$

consequently,

$$I_2 / m_a < n < I_2 / m_c$$

The inequality sign should be changed for deceleration.

As a rule, if an acceleration or a deceleration of the pulsation on I_2 is desired, m_b modulus may be generated in the following form:

$$m_b \text{ mean} + x_i \Delta \qquad (i = 1, 2, ..., n)$$

where $\Delta = m_a - m_c$.

x_i may be arbitrarily chosen in order to create the law L, which gradually transforms the constant tempo of a pulsation into another different constant tempo.

In order to avoid discontinuities at the contact with m_a and m_c, i.e., in order to observe the conditions $m_{b1} < m_a$ and $m_{ba} > m_c$ (for acceleration), and $m_{b1} > m_a$ and $m_{ba} < m_c$ (for deceleration), x_i should fulfil the following two conditions:

1° $\sum_{i=1}^{n} x_i = 0$

2° $x_1 > \dfrac{m_a - I_2/n}{m_a - m_c}$;

$x_n < \dfrac{m_c - I_2/n}{m_a - m_c}$ (for acceleration)

The inequality sign should be changed for deceleration.

6.3.3. Case of structures with degree of complexity n, $n > 1$

The laws of the distribution of real and virtual pulsations in a polymodular time structure of degree $n > 1$ are, in principle, the transposition of the laws concerning structures of degree 1 in the structures of higher degrees, in the sense that, in a given collection C_{n-1} of reference systems ($[TRS]^{n-1}$), in the pairs of systems $C\ [TRS]^{n-1}$, $[TRS]^n$ the laws for the distribution of pulsations in $[TRS]^n$ are identical with those specified for TRS under item A (where $TRS \rightarrow [TRS]^n$ and $M \rightarrow M^n$), and the laws of distribution in the $[TRS]^{n-1}$ systems are identical with those those specified for the RPS systems under item B (where $RPS \rightarrow [TRS]^{n-1}$ and $M \rightarrow M^{n-1}$).

6.4. POLYMODULAR TIME AND CHRONOLOGICAL TIME

6.4.1. Chronological time viewed as a time reference system $[TRS]^{n+1}$ for a polymodular structure $[PMT]^n$

A polymodular time structure of a certain degree of complexity, $[PMT]^n$, may be expressed as a function of the chronological time, GMT, by associating to its time reference system of the highest degree, $[TRS]^n$, a time reference system $[TRS]^{n+1}$. $[TRS]^{n+1}$ represents the chronological time. By definition, the chronological time has a constant tempo and the time module becomes one of its measuring units (seconds, minutes, etc.).

In practice, $[TRS]^{n+1}$ may be avoided by directly expressing in $[TRS]^n$ the values of the time modules M in values of chronological time.

Fig. 6.15 represents a numerical example of a polytemporal structure referred to as the chronological time. The pulsations that develop on RPS_1 and RPS_3 have constant tempi defined with respect to the minute

$$\left(\sum_{i=1}^{5} M_i = 60''\right).$$

The pulsations developing on RPS_2, and on the intervals I_1 and I_3, in RPS_4, have also constant tempi, expressed with respect to the various intervals in TRS. The law L_1 operates in the sense that over the nine m_b units, the value $m_a \in 1$, decreases gradually towards the value of $m_c \in I_3$. The law L_2 operates in the sense that it establishes the moment the pulsations of the RPS_5 ($M_1 + 4''\ 7$)

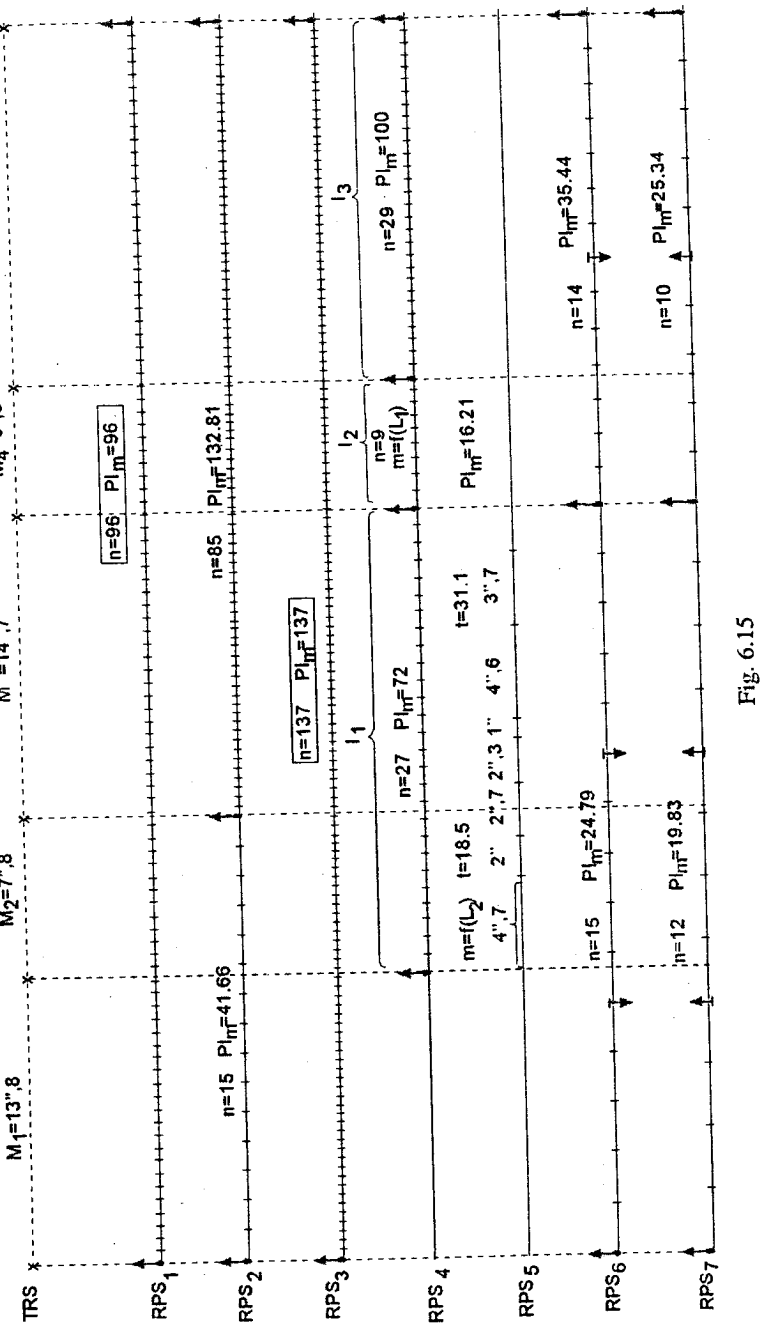


Fig. 6.15

occur, the value of the m intervals, defined on the first five pulsations, as well as the constant value these intervals have over the remainder of the system. The RPS_6 and RPS_7 pulsations have different constant tempi too, exhibiting a number of spontaneously synchronized pulsations, at the level of the two systems.

6.5. TOPOLOGICAL ASPECTS OF THE POLYMODULAR TIME

In a topological sense, the pulsations defining modules are "points". We may define a distance-function on the set of points representing pulsations as follows:

The distance between two given points is the time interval defined by the pulsations represented by the two points. This distance-function defines a *pseudo-metric space*, since the distance between two different points may be zero (when the two points belonging to some different pulsation systems are synchronous).

The change of the law of distribution for the pulsations of a pulsation system is a topological mapping, since the topological characteristics (neighbourhood and continuity) are preserved and the mapping is bijective and bicontinuous.

For example, an acceleration or a deceleration of the pulsation tempo of a pulsation system may be expressed by a transformation through homothety,[7] the pulsations being represented as a set of colinear points displayed in a plane on a time axis (Fig. 6.16).

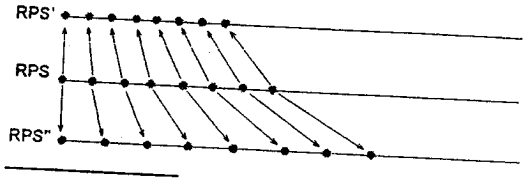


Fig. 6.16

[7] Recall that homothety is an affine transformation defined by the following equations:

$$x' = x_0 + k(x - x_0), \quad y' = y_0 + k(y - y_0), \quad k \neq 0$$

The point C_0 of the coordinates x_0, y_0 is a fixed point in this transformation, and it is reffered to as the *homothety center*. The number denoted by k is called the homothety ratio and represents the C_0M'/C_0M ratio of the C_0M' and C_0M segments, defined by the homothety center C_0, by a point M in the plane and by the image M' through homothety. The points C_0, M and M' are colinear (Fig. 6.17).

Fig. 6.17

Homothety being a topological transformation, the acceleration and deceleration of the pulsations tempo in a pulsation system (represented as a set of colinear points in a plane) can be interpreted as topological transformations; it is easy to see that *any transformation* on the law of distribution of these points induces a topological mapping of the image of that pulsation system, since the concepts of neighbourhood and continuity are preserved (Fig. 6.18).

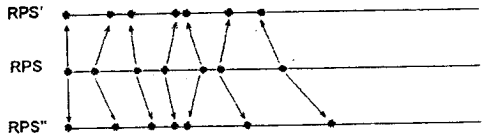


Fig. 6.18

The above reasoning may lead to the conclusion that the image of the time structure of a finite number of repetitive events can present, in practice, an infinite number of forms, each of them representing the topological mapping of the same time structure. Consequently, the time structure of a finite number of repetitive events can be in continuous change, remaining, however, invariant in essence, as far as the topological characteristics are concerned.

6.6. THE POLYMODULAR TIME STRUCTURE OF MUSIC

a) traditional music: polymodular time structure of 1st degree of complexity;

b) polytempi music: polymodular time structure of degree of complexity n, $n > 1$.

6.6.1. Traditional music

In the sense of the sentence (a), the "beats", expressed by the gestures of the conductor or by the rhythmically uttered sound symbols of a row of numbers, constitute the virtual pulsation (*VP*) of a time reference system (*TRS*). The appearance of musical sound represents the real pulsations (*RP*). A row of musical sounds ordered in time constitutes a real pulsation system (*RPS*).

Let us consider the last four bars of this page of George Enescu's First Symphony (Fig. 6.19)

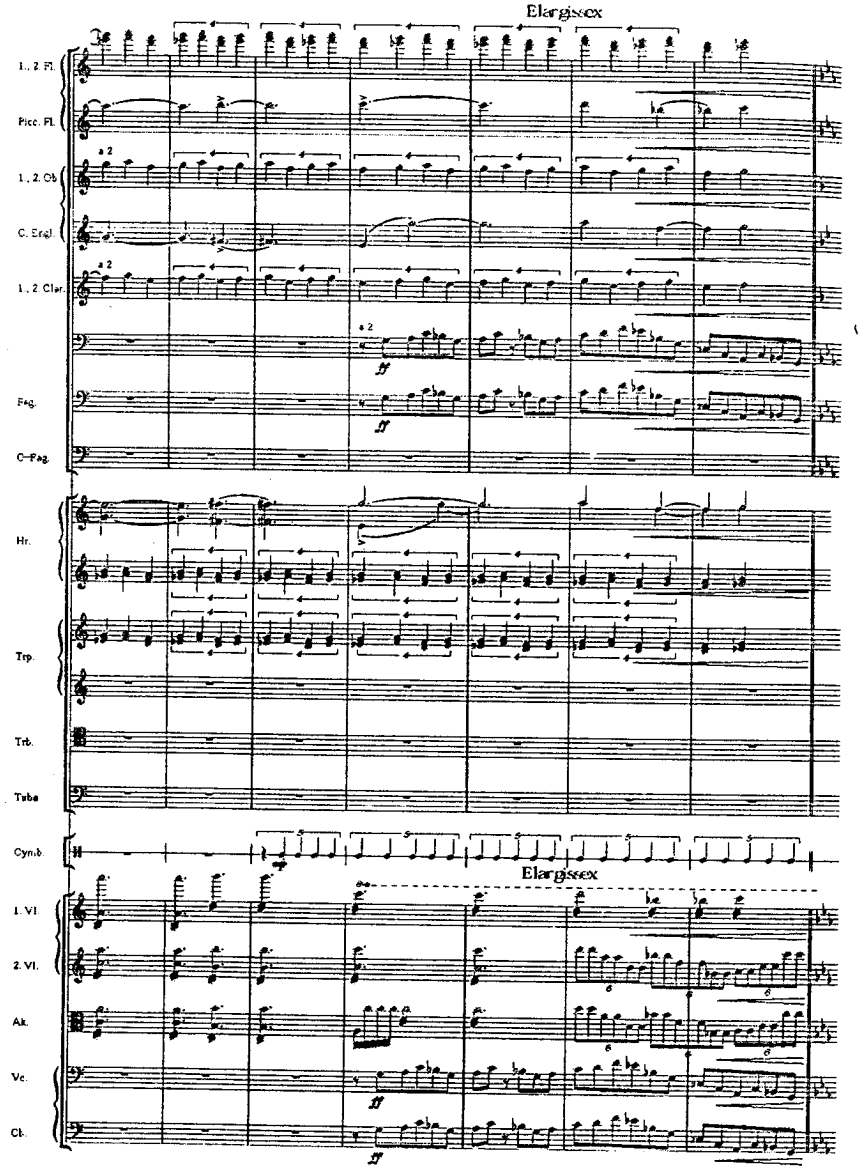



Fig. 6.19

The polymodular time structure of this fragment can be expressed by the diagram (Fig. 6.20).

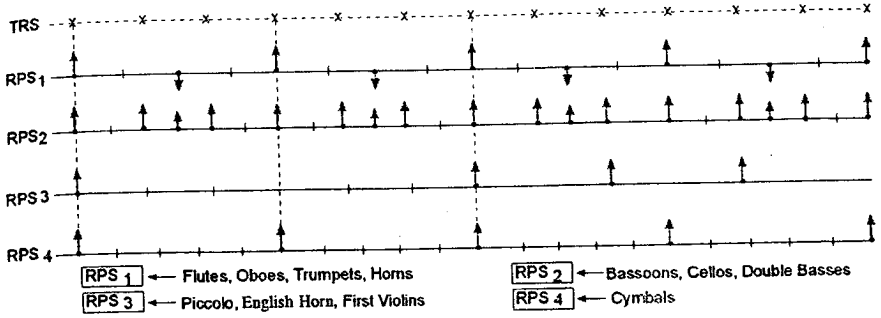


Fig. 6.20

The in-time occurrence of the sounds played by the different groups of instruments is represented by a collection of four real pulsation systems (*C*). The beats of the conductor are represented by the virtual pulsation system (*VPS*).

According to the introduced definitions, we are faced with a polymodular time structure. All real pulsations of the structure are controlled by the same single time reference system; hence, the structure has a first degree of complexity. The judgement may be extended to all traditional musical structures.

A possible hierarchied organization of musical structures depending on the degree of coherence of the various systems of real pulsations representing musical sounds.

In traditional music, the concept of degree of coherence can be useful for determining the extent to which the sensation of tempo in certain played musical structure is induced in the listener.

For example, when hearing the fragment given in Fig. 6.21, the listener is not aware of a tempo in the structure, as none of the reference system virtual pulsations is marked by a real pulsation (sound). The diagram of the polymodular time structure of this fragment is similar to that in Fig. 6.3. The fundamental pulsation system is totally virtual (*TVF*).

On the other hand, the baroque music in the manner of J. S. Bach, or the jazz music, easily allow the perception of the respective tempo, just because there is a fundamental system of totally real (Fig. 6.22) or partially real pulsations, but with great local coherence, marking the tempo of the reference system pulsations *VPS* and by this including, in those hearing it, the value of the played music measuring unit. As a rule, such structures prevail in the music of all times. This is illustrated in Fig. 6.22 with Beethoven's *Quartet* op. 133, and in the diagram of the polymodular time characterizing it (*TRF*).

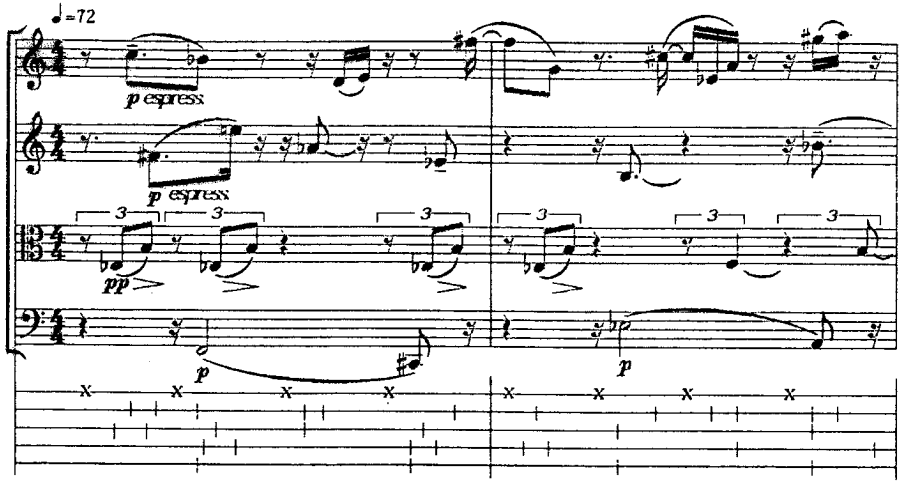



Fig. 6.21

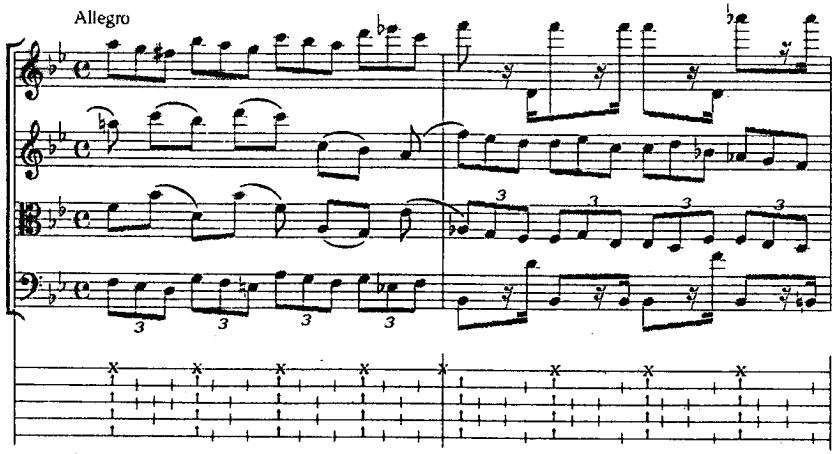



Fig. 6.22

We might say that, statistically, the public on the whole has a greater adherence for musics with a higher degree of coherence at the pulsation system level, i.e. they are more delighted to hear musics whose tempo is obviously discernible. As a result, a possible hierarchical form might be taken into consideration for musical structures depending on the value of the systems of fundamental pulsations characterising them. This hierarchical form would induce the degree of accessibility of the respective structures.

When, for the first degree of complexity, the degree of coherence of the *PMT* structures has an analytical significance, we shall further see that, for higher degrees of complexity, the degree of coherence is important as far as a like manner of music composing is involved.

6.6.2. Polytempi music

The polymodular time concept with degree of complexity higher than 1, can, in a most natural way, theoretically define a general model of polytempi musical structure. *In fact, the problem of polytempi music amounts to the problem of structuring the pulsations of several different time reference systems (TRS). As was already shown, from the polymodular time theory point of view, the pulsations ("beats") defining the value of the measuring unit ("meter") of the monodic or polyphonic traditional music may be considered as a system of virtual pulsations* (VPS), namely, a time reference system (*TRS*).

Consequently, *the polytempi music may be considered as having a polymodular structure of degree n, n>1, where the different reference systems TRS are structured by means of some hypersystems* $[TRS]^2$, $[TRS]^3$, ..., $[TRS]^n$.

The application of the polymodular time concept to music heads to the notion of "*polymodular time music*", or "*music in structural polytempi*".

According to the above-mentioned, when in a given interval from $[TRS]^n$, defined by a certain number of modules M^n, the modules have a constant value (T_p^n constant), such a module M^n represents the measuring unit, the general "meter" of the respective musical structure.

It has been pointed out that, in most traditional music, the tempi of the emergence points, of the different sounds are defined by the division by a positive integer of the time interval defined, in its turn, by a "meter", or by several such measuring units. The same general law may be applied to a music written in structural polytempi at the level of the different $[TRS]^n$ ($n > 1$).

The different reference systems of different degrees $[TRS]^j$ ($j = 1, 2, ..., n$) may be interpreted as being the "beats" of some imaginary conductors, a certain such conductor corresponding to each $[TRS]^j$ of a certain degree. Since a $[TRS]^j$ ($j > 1$) may be considered as a "reference system of the reference systems of a lower degree", from the musical point of view, such an imaginary conductor of degree j ($j > 1$) may be equally considered as being "*a conductor of the conductors of lower degrees*" (see, in this connection, the tree-like organization. The "beats" of the imaginary

conductor of degree n, defining the time modules M^n, and, hence, the tempo T_p^n, represents the reference system that organises the time of the whole musical structure conceived in polymodular time.

According to foregoing definitions, the set of synchronous virtual pulsations (*SVP*) can induce into a score written in polymodular time, both at the general level $[PMT]^n$, i.e. at the level of the time reference systems $[TRS]^n$, and at that of the different substructures $[PMT]^{n-1}$, $[PMT]^{n-2}$, ..., *PMT*, different degrees of coherence, i.e. *totally virtual, partially virtual, partially real* and *totally real* fundamental pulsation systems.

6.6.3. Music with polymodular time structures of degree of complexity 2

The polymodular time structures of degree of complexity 2 already offer a very convenient mode of organizing a number of time reference systems *TRS* by means of a general virtual pulsation system, a reference hypersystem $[TRS]^2$.

Fig. 6.23 shows a polymodular time structure of complexity degree 2 $[PMT]^2$. At the level of the *TRS* systems, the structure has the first degree of coherence (Totally Virtual Fundamental Pulsation System).

A collection of *RPT* is associated to each *TRS*. The real pulsations are directly represented by the occurrence of musical sounds produced by an organ, a violin and percussion instruments. To have the best clarity possible, some *RPS* collections have only one element each (a single "voice"). For a purely illustrative sake, the *TRS* system pulsations are marked on the score (Fig. 6.24).

Of course, in the ordinary process of writing a score conceived in polymodular time, there is no need to represent these pulsations, as in the monotempo traditional music there is no need to represent the "beat" of the measuring unit.

Fig. 6.25 represents the same polymodular time structure, but this time the real pulsations are represented by the occurrence of the sounds of a trio (clarinet, bassoon, violin). For the sake of illustration, the in-time occurrence of the "beats" of the different *TRS* are marked.

Fig. 6.26 shows another fragment of the above-mentioned trio, having at the level of the *TRS* a third degree of coherence (Partially Real Fundamental Pulsation System), which is explicitly represented in Fig. 6.27.

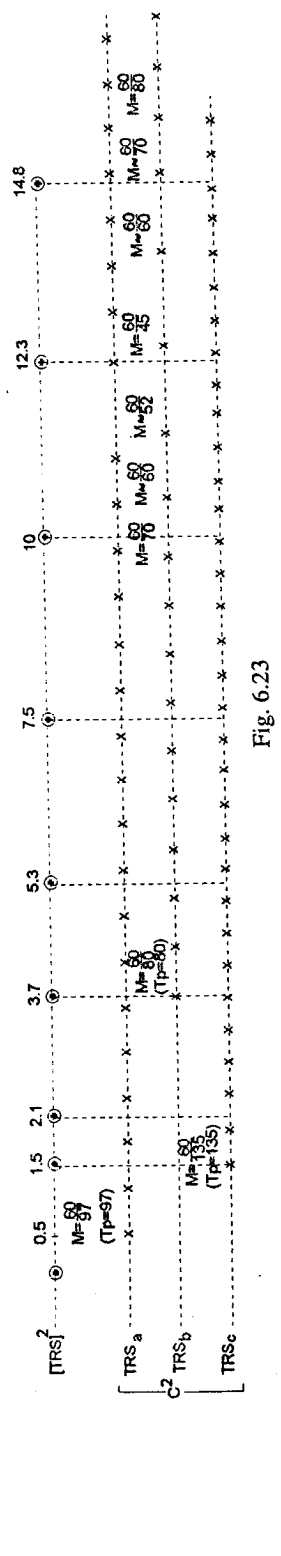


Fig. 6.23

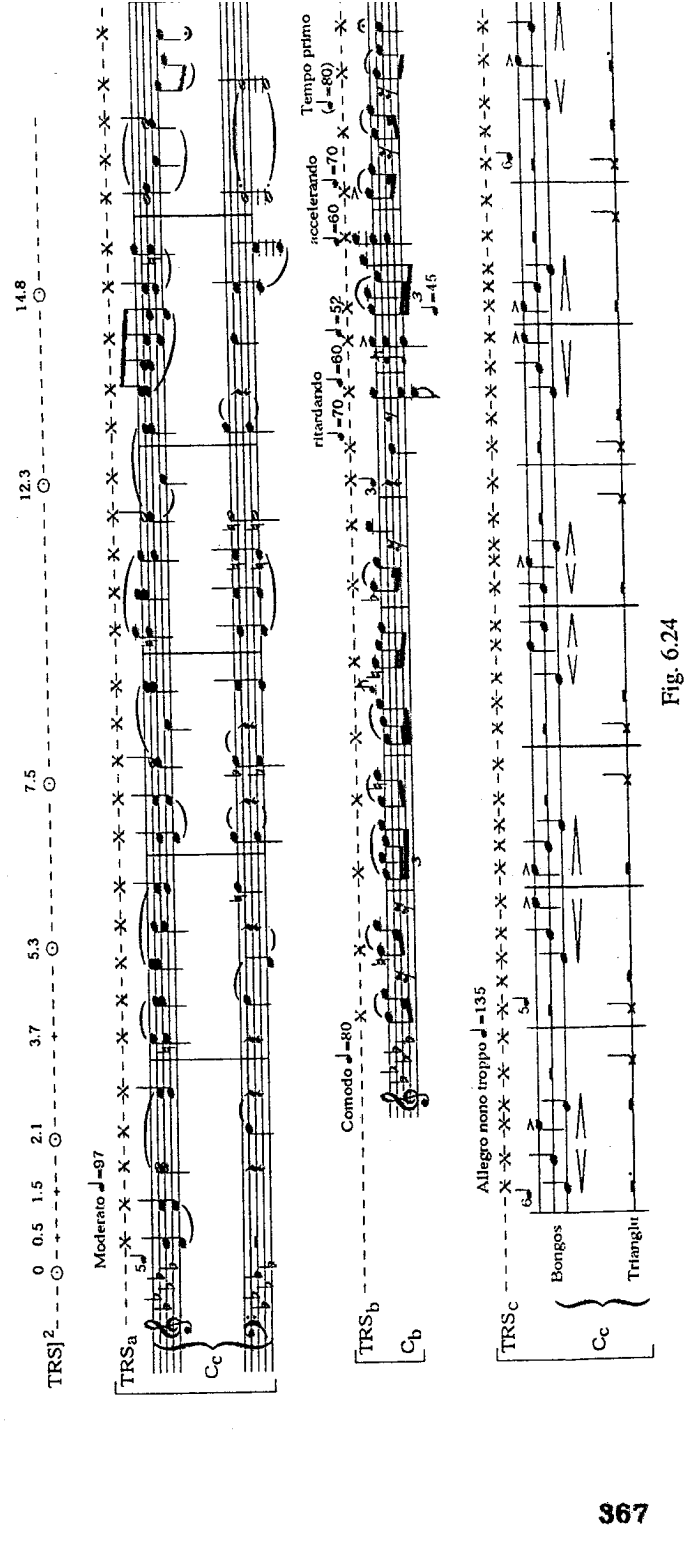


Fig. 6.24

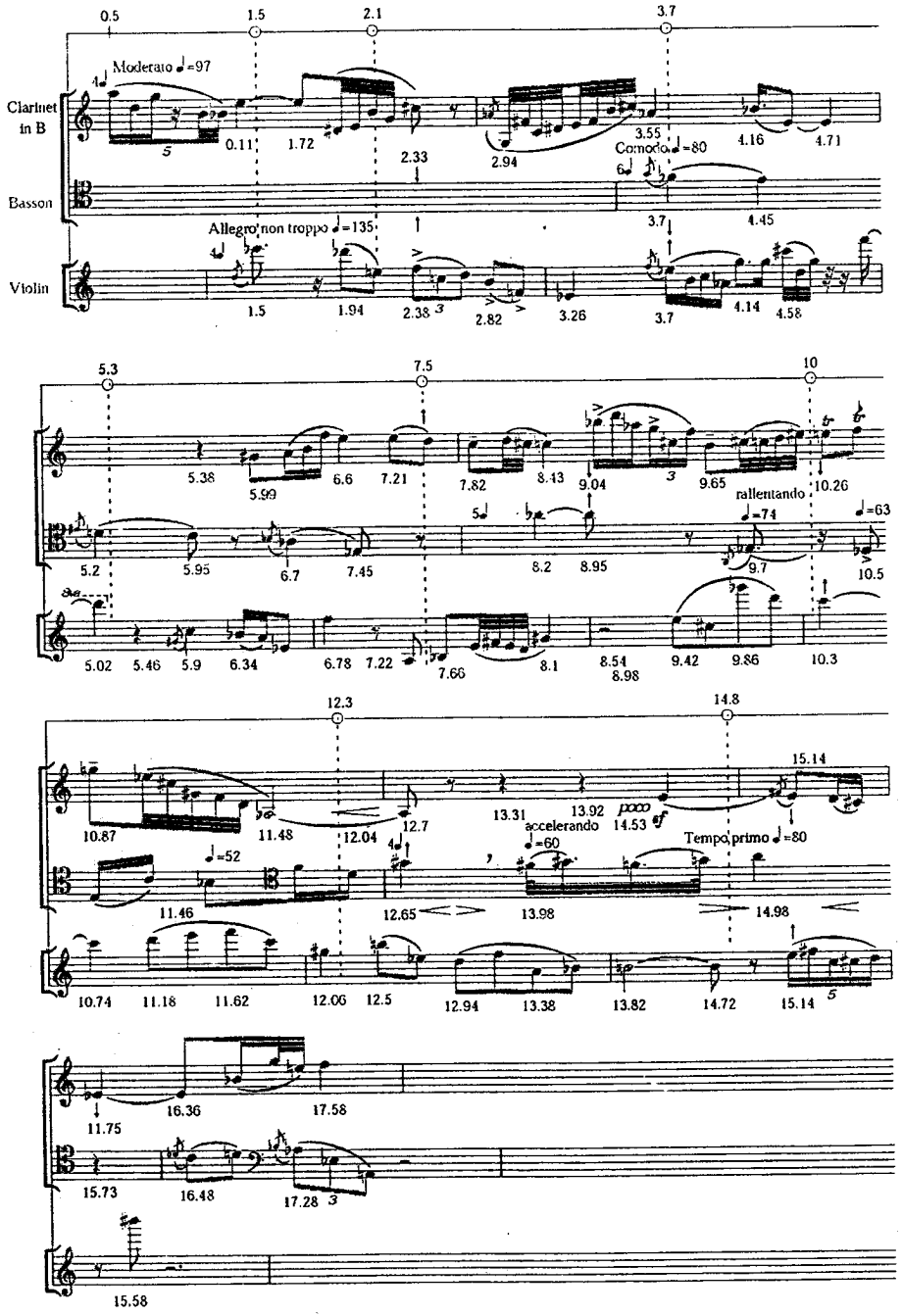



Fig. 6.25

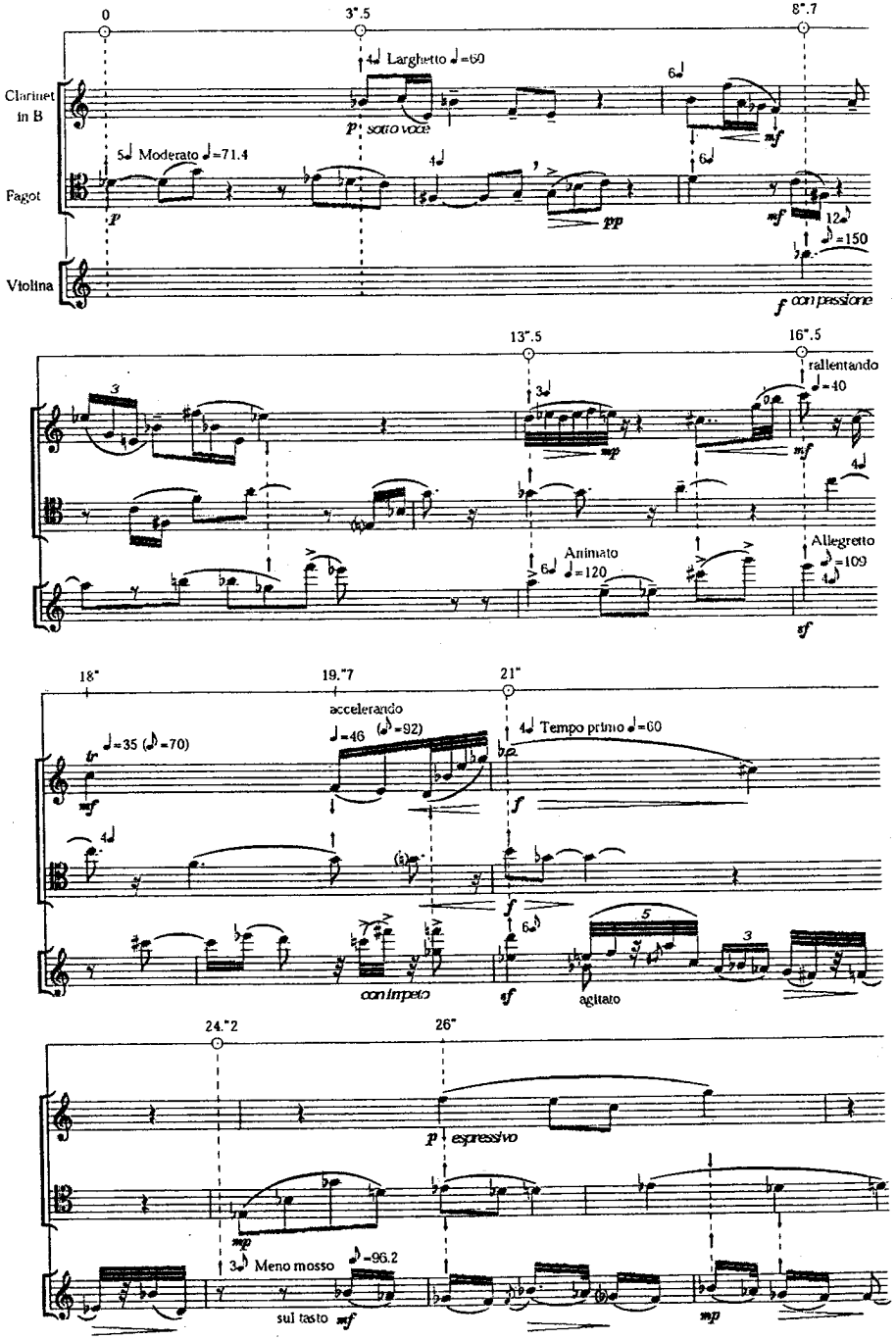


Fig. 6.26

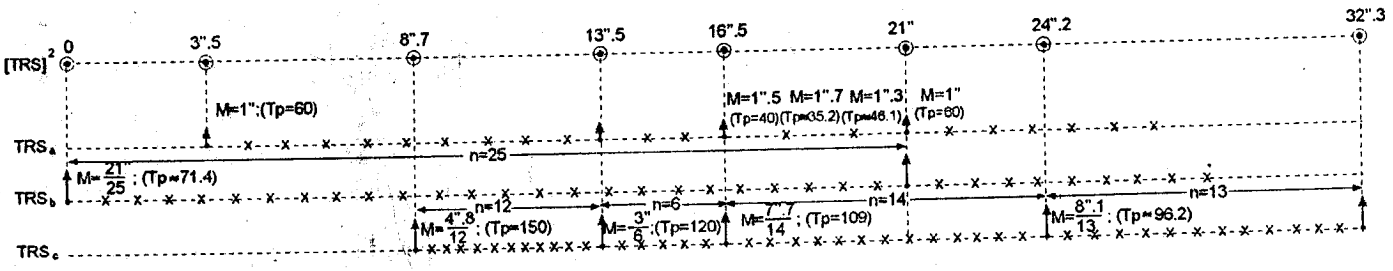


Fig. 6.27

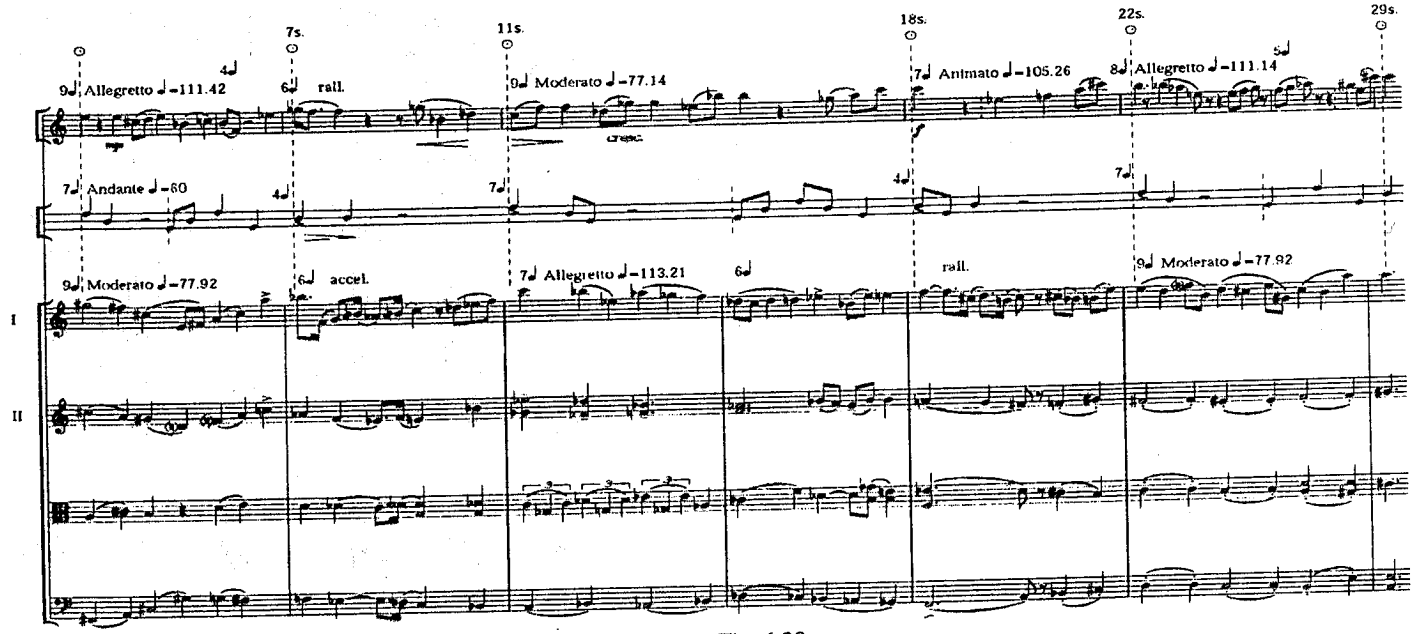


Fig. 6.28

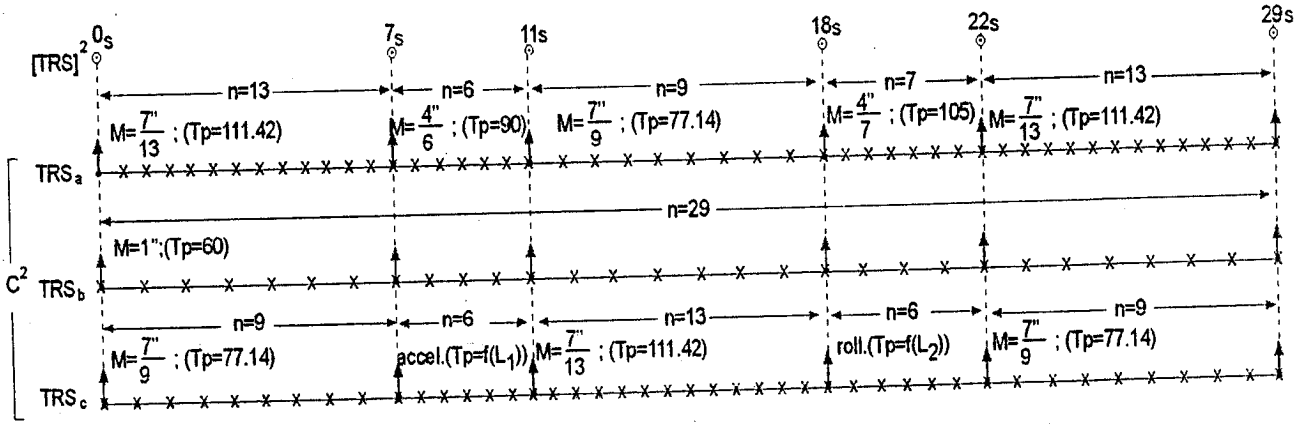


Fig. 6.29

The degree of coherence of the collection of reference systems *TRS* and the system $[TRS]^2$ induced by the characteristics of the $[FPS]^2$, plays an essential part in the possibility offered the listener to discern the general tempo of such a temporal structure. The totally real fundamental pulsation system of second degree $[TPS]^2$ seems to bee the most adequate for the listener to perceive the structure of a musical fragment conceived in $[PMT]^2$ and its general fundamental tempo in particular. The score fragment (Fig. 6.28) and its time diagram (Fig. 6.29) illustrate a collection of *TRS* having degree of coherence IV. In the example shown, TRS^c is predominant over a collection of four systems of real pulsations representing a string quartet.

If the collection of *TRS* and $[TRS]^2$ is provided with a Totally Real Fundamental Pulsation, the tempo of the pulsations of $[TRS]^2$ has a constant value (isochronous M^n) and the different *TRS* develop, between two consecutive pulsations of $[FPS]^2$, a constant number of identically organised pulsations. Then, the $[PMT]^2$ presents a recurrent, cyclical structure at the level of the respective *TRS*. This type of structure represents the transposition at the level of the $[PMT]^2$ of the *PMT* structures characteristic for some music described before, where a fundamental pulsation system with a steady, constant tempo can be detected.

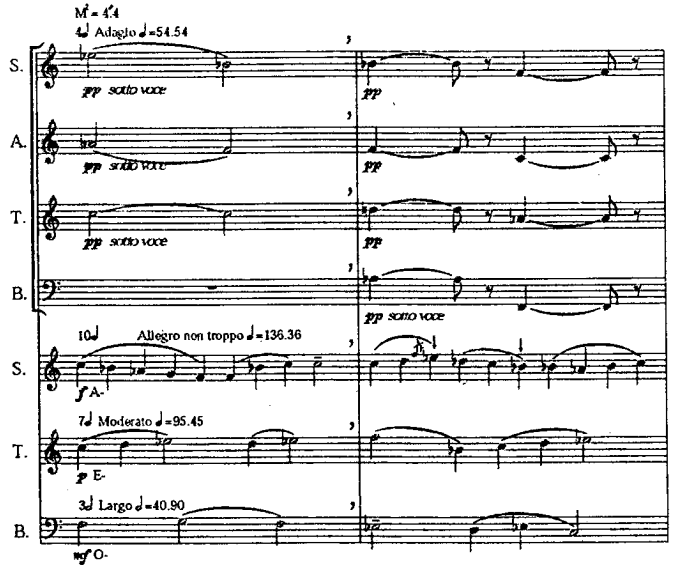



Fig. 6.30

Fig. 6.30 shows a fragment of the author's "Interpolated Choral in Byzantine style" conceived in structural polytempi $[PMT]^2$ for four choral

groups presenting, at the level of the different time reference systems, a Totally Real *FPS* endowed with a cycle organisation. Fig. 6.31 represents its polymodular time diagram.

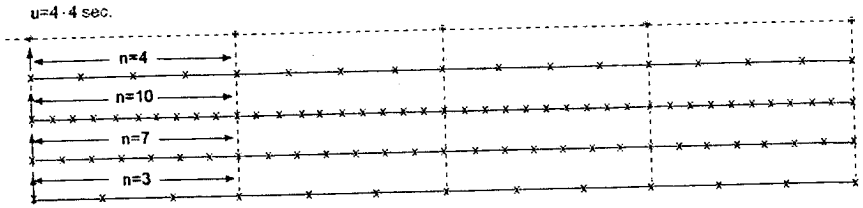


Fig. 6.31

Although the problem of layered music endowed with different simultaneous tempi can be fairly well solved by polymodular time structures of second degree of complexity, in order to be theoretically consistent, yet considering also the unusual aspect of such a score, we give in Fig. 6.32 a musical fragment conceived in *PMT* (polymodular time structure of third degree of complexity).

6.7. POLYTIMER

The music written in polymodular time raises, of course, special difficulties for the interpreters and, especially, for the conductors, since, for a proper performance, each "voice" (or group pf voices having the same tempo) supposes not only its own conductor, but a strict coordination and hierarchy organization too.

The problem of playing such music in rehearsals and in concerts was solved by using the device, referred to as Polytimer, patented by the author in various countries in 1970. It was built as a prototype at the Syracuse University in Syracuse, N.Y., USA, by professor Edward Stabler, in 1972.

The polytimer is a computer that can be programmed to produce simultaneous signals of different tempi, according to scores conceived in polymodular time structures. The signals are transmitted to the performers (musicians, actors or dancers) *via* light and sound signals *qua "beeps"*, which are received in miniature ear phones. The signals are transmitted and received by wires or by radio using appropriate channels.

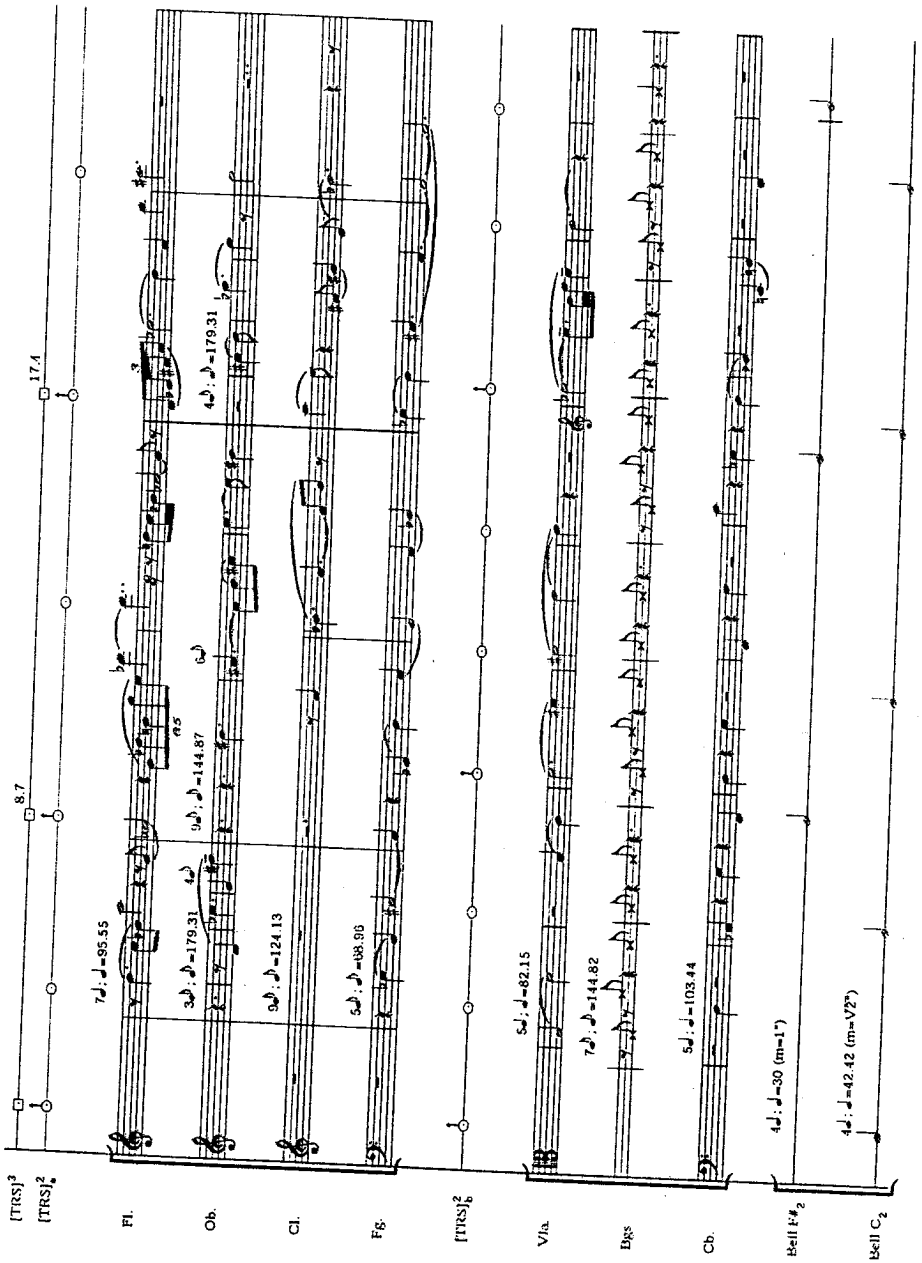


Fig. 6.32

The polytimer ensures a strict coordination of signals which indicate the various tempi used in the music and stage movement. The tempo of the signals transmitted on each channel may assume any value, may be changed at any point as demanded by the score, and transmissions from a certain tempo value to another value can be provided by accelerations or decelerations while the over-all structures continue to be strictly organised and controllable at any time instant.

The polymer permits the execution of partially tape-recorded and partially performed (by instruments and voices) scores while the suggested temporal structure is strictly obeyed.

The polytimer is the first musical instrument conceived to generate time structures in concerts and is a natural consequence of the theory regarding the polymodular time. The successive models have increased its performances. The latest model, carried out by the Romanian engineer Ion Predonescu and his team, is serially manufactured by the ITC company in Bucharest.

ADDENDA

A.1. INTRODUCTORY CONCEPTS OF TOPOLOGY

A.1.1. Topological spaces, mappings and invariants

A *topological space* is a couple of objects $<X, \tau>$, where X is a set of elements (usually referred to as *points*) of a non-specified type, and τ is a family of parts of X having the following properties:

1. X and the empty part ϕ of X belong to τ;
2. any union of sets from τ belongs to τ;
3. the common part of two sets from τ belongs to τ also.

X is referred to as *support* of the topological space, and the sets in τ are referred to as *open sets*. The family τ is a topology on the set X. The complement relative to X of an open set is, by definition, a *closed set.*

A point x being given in the topological space $<X, \tau>$, the *neighbourhood of* x is called any set A when between x and A an open set may be interposed; in other words, there is a set G in τ, so that x belongs to G, and G is included in A. A neighbourhood of the point x is usually denoted by V_x or $V_{(x)}$ to show its dependence on the considered point x.

A simple example of topological space is the $\mathbf{R}^3$, τ_0 couple, where $\mathbf{R}^3$ is the set of points in the Euclidean tridimensional space, and τ_0 is the so-called *usual topology* of the Euclidean space, defined as follows: a set G of points from $\mathbf{R}^3$ is open when from every x in G there is a sphere centred in x and included in G. It is easy to verify that the family τ_0 of the sets defined in this way satisfies properties (1), (2) and (3), hence constitutes (makes up) a topology on $\mathbf{R}^3$. In this topology, the neighbourhoods of a point x are all the sets $V_{(x)}$ with the property that there is a sphere centred in x and included in $V_{(x)}$.

Two sets X and Y being given, a *function* or *mapping* of X in Y may be defined as a system of three objects $<X, Y, f>$, where f is a rule associating to every element in X a single element, determined from Y. Let us note by $f_{(x)}$ the element associated to point x from X. The mapping X, Y, f is also noted as $f: X \to Y$.

It is said that the mapping $f: X \to Y$ is *injective* when for any pair of distinct elements x and y from X the associated elements $f_{(x)}$ and $f_{(y)}$ are distinct also.

A set A contained in X being given and a mapping $f: X \to Y$, let us note by $f(A)$ the set of all the elements y from Y, for which there is an element x in X, so that $f(x) = y$. The set $f(A)$ is referred to as the *image* of A through f.

It is said that the mapping $f: X \to Y$ is *bijective* or that it is a *bijection* between X and Y, when it is an injective mapping and if, in addition, we have $f(X) = Y$.

When a mapping $f: X \to Y$ and a set $B \subset X$ are given, let us note by $f^{-1}(B)$ the set of those points x in X for which $f(x)$ belongs to B. The set $f^{-1}(B)$ is the *reverse image* of B through f.

Let $f: X \to Y$ be a bijective mapping, we may define the *reverse mapping* $f^{-1}: Y \to X$ as follows: for any element y in Y we consider $f^{-1}(y) = x$, where x is that (uniquely determined) element from X for which $f(x) = y$.

Two topological spaces $<X, \tau>$ and $<Y, \sigma>$ being given, the mapping $f: <X, \tau> \to <Y, \sigma>$ is said to be *continuous* on X when for any open set in $<Y, \sigma>$ its reverse image through f is an open set in $<X, \sigma>$. The simplest example of continuous mapping of a topological space in itself is the identical mapping of this space, i.e. the mapping which associates this very element to every element in X.

Let $<X, \tau>$ and $<Y, \sigma>$ be two topological spaces: $f: <X, \tau> \to <Y, \sigma>$ is said to be a *topological mapping* when it is continuous on X, bijective, and its reverse $f^{-1}: <Y, \sigma> \to <X, \tau>$ is continuous on Y. It is easy to see that the reverse of a bijective mapping is also a bijective mapping. There results that a topological mapping is equivalent to a bijective and bicontinuous mapping (in other words, continuous together with its reverse).

Let $<X, \tau>$ and $<Y, \sigma>$ be two topological spaces. $<Y, \sigma>$ is said to be the topological mapping (transformation) of $<X, \tau>$ and $<Y, \sigma>$ is obtained from $<X, \tau>$ through a topological mapping (transformation) if there is a topological mapping $f: <X, \tau> \to <Y, \sigma>$.

A topological mapping between two topological spaces is usually referred to as a *homeomorphism* between these spaces. It is also said that the respective spaces are *homeomorphic.*

It easy to see that, in the set of topological spaces, the relation of topological transformation is reflexive: any topological space is its topological transform, symmetrical (if $<Y, \sigma>$ is the topological transform of $<X, \tau>$, then $<X, \tau>$ is the topological transform of $<Y, \sigma>$) and transitive (if $<Y, \sigma>$ is the topological transform of $<X, \tau>$ and $<Z, \omega>$ is the topological transform of $<Y, \sigma>$, then $<Z, \omega>$ is the topological transform of $<X, \tau>$).

We recall that a binary reflexive, symmetrical and transitive relation in a set A is said to be an *equivalence relation* in A. Hence, we may say that the

relation of topological transformation is an equivalence relation in the set of topological spaces. An equivalence class in terms of this relation consists of all the topological spaces obtained from a given topological space through a topological transformation.

Let P be a property of a topological space or of a set in a topological space. We say that P is a *topological property*, when it is kept up through any topological transformation applied to the considered space or considered set. It is also said about a topological property that it is a *topological invariant*. An example of topological property is the property of *connectivity*. We say that a topological space is *connected* when its decomposition in two non-null, disjoint and, at the same time, closed and open sets is impossible. The connectivity property is a mathematical description of the intuitive idea of "all of a piece". That is, the set in Fig. 1a is connected, while the set in Fig. 1b is not connected.

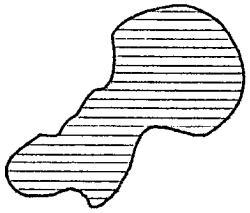


Fig. 1a

Fig. 1b

In the given examples, the connectivity and the non-connectivity of the set are intuitive facts, directly noticeable. But, there are more complex situations also, where the "all of a piece" character is less evident. Here is an example of this type. Let us consider in the plane xOy the set obtained as union of the graph of the $h(x) = \sin 1/x$ function, defined on the interval 0,1 and the line segments on the Oy axis and included between the ordinates -1, 1/2 and 1/2, 1, respectively (Fig. 1c).

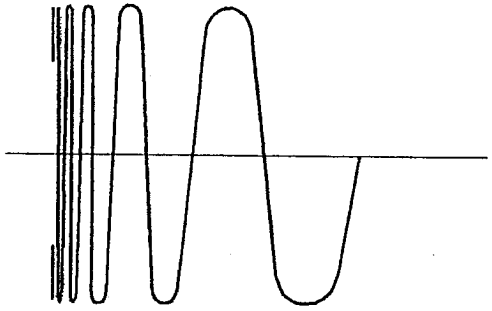


Fig. 1c

It may be proved that, when a topological space is connected, then any topological space obtained from the first one through a topological transformationn is also connected. Hence, the connectivity is a topological invariant. We may define the connectivity property not only for the entire space, but also for a part of it, as follows. Let A be a part of the topological space $\langle X, \tau\rangle$. We define the *relative topology* to A of this space as the totality of form sets $A \cap \sigma$, where $\sigma \in \tau$. The relative topology to A is noted by $\tau \cap A$. Any set from this topology is referred to as *open set* on A. The complement in connection with A of an open set on A is a *closed set* on A. We say that a part of A of a topological space $\langle X, \tau\rangle$ is connected when there is no decomposition of A in two non-null, disjoint and at the same time closed and open sets on A. The property of connectivity of a set A from the $\langle X, \tau\rangle$ space is a topological invariant. It means that, denoting by $\langle X', \tau'\rangle$ a topological space obtained from $\langle X, \tau\rangle$ through a topological transformation and by A' the image of A through this transformation, A' will be a connected part of $\langle X', \tau'\rangle$.

Another important example of topological invariant is the notion of topology itself. This means that, if X' is the image of X through t, and the family τ' of sets from X' is the image of τ' through t, then τ' is a topology in X'.

Another example of topological invariant is the property of a set to be a neighbourhood. Hence, when V is a neighbourhood of the point x in the space $\langle X, \tau\rangle$ and if $\langle X', \tau'\rangle$ is the topological transform of $\langle X, \tau\rangle$ through a topological transformation t, then the image V' of V through t is a neighbourhood in $\langle X', \tau'\rangle$ of the image $t(x)$ of x through t.

A.1.2. Compound transformations

Let h be a homeomorphism transforming the set A in the set B, and let g be a homeomorphism transforming the set B in the set C.

The transformation product $g \circ h$ is defined as being the mapping leading to A in C according to the following law: $(g \circ h)(x) = g(h(x))$, for any $X \in A$. The composition of two homeomorphisms is a homeomorphism,

A.1.3. Topological transformation of functions

A real function F being given, as defined on the interval a, b, a topological transform of f is defined as follows. Let us consider on the interval $[\alpha, \beta]$ a function φ continuous and strictly increasing, for which $\varphi(\alpha) = a$, $\varphi(\beta) = b$. (The property of being strictly increasing consists, by

definition, in the fact that, for $t' < t''$, we have $\varphi(t') < \varphi(t'')$.) It can be proved that any homeomorphism between two intervals of a line (here also including the case when the two intervals coincide) is a continuous, strictly increasing mapping of one of the two intervals on the other one. A topological transform of the function f is, by definition, a function g defined by a relation of the form $g(t) = f(\varphi(t'))$, where φ is a continuous and strictly increasing mapping of $[\alpha, \beta]$ on $[a, b]$. We can also define a topological transform of f by means of a homeomorphism φ from $[a, b]$ to another interval $[\alpha, \beta]$, as being that mapping g off $[\alpha, \beta]$, which satisfies the relation $g(\varphi(x)) = f(x)$ for any x from $[a, b]$. The difference between these two modes of defining a topological transform of a function is unessential, since the topological transformation is an equivalence relation.

The topological transformation of a function is a particular case of a more general topological transformation, that of a curve. It will be necessary to first mention the notion of curve.

Let us consider a continuous mapping of the compact interval (i.e. closed and bounded) [a.b] in the n-dimensional Euclidean space $\mathbf{R}^n$. Such a mapping f: $[a, b] \to \mathbf{R}^n$ is, by definition, a curve in $\mathbf{R}^n$. Let φ be a homeomorphism from the interval [a, b] in the compact interval $[\alpha, \beta]$ and let g: $[\alpha, \beta] \to \mathbf{R}^n$ be a curve, so that $f(x) = g(\varphi x))$for any x in $[a, b]$. Under these conditions we say that the passage from the curve f to the curve g is obtained through the agency of the homeomorphism φ or that the two curves are topologically equivalent. Thus, the curves are divided into topological classes of equivalence. The above mentioned particular case is obtained from the general notion of topological equivalence of two curves, considering $n = 1$.

We will agree to consider that two curves obtained from one another through a topological transformation are undiscernible. As a matter of fact, it can be proved that two such curves have the same image and induce the same order of the points on this image. But, the converse is not true. It is possible that two curves may have the same image, without being topologically equivalent. Thus[1], the curves

$$C_1: \begin{matrix} x = t \\ 0 \le t \le 1 \end{matrix} \qquad C_2: \begin{matrix} x = f(t) \\ 0 \le t \le 1 \end{matrix}$$

where

$$f(t) = \begin{cases} \sin 1/t & t \neq 0 \\ 0 & t = 0 \end{cases}$$

[1] For details, see S. Marcus, *Noţiuni de analiză matematică*. Editura Ştiinţifică, Bucureşti, 1967, p. 76.

have the same image, i.e., the segment of line with the extremities (0, 0) and (1, 2). But, it can be proved that these curves are not topologically equivalent, besides an intuitive fact, since, while with the C_1 curve, the image is traversed in a single sense, when t varies from 0 to 1, with the C_2 curve, the image is traversed with an infinity of advances and returns.

The general concept of curve, mentioned above, includes as particular cases the curve in plane (case $n = 2$) and the curve in tri-dimensional space (case $n = 3$). Usually, a curve in n-dimensional space is not directly given through a mapping, such as the above f mapping (also as vectorial mapping), but by the n scalar components of f, such as $f_1, f_2, ..., f_n$. In fact, the curve is given by a parametric representation

$$x_1 = f_1(x), x_2 = f_2(x),..., x_n = f_n(x)$$

where $x_1, x_2, ..., x_n$ are the coordinates of the point $f(x)$ in $\mathbf{R}^n$.

A.2. INTUITIVE INTRODUCTION OF SOME FUNDAMENTAL TOPOLOGICAL CONCEPTS

A.2.1. Neighbourhood, continuity, connectivity, bicontinuous and one-to-one transformations

When we refer to sets of unspecified elements, the fundamental topological concepts are difficult to present in an intuitive way. In exchange, when the elements of the considered set are points of a geometric space, for instance a plane, the topological concepts acquire an easily perceptible image through reference to the direct reality.

Thus, the notion of neighbourhood of a point is defined by the usual meaning of the word, i.e. by the set of points in the neighbourhood of the given point, of the points "surrounding" it. A neighbourhood of radius r of a point P is readily visualized as being the circle with centre in P and of radius r (Fig. 2).

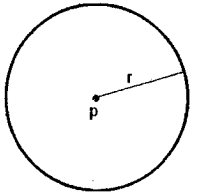


Fig. 2

The concept of continuity is easily understood, also, in the case of points in a plane. Here is a simple example. Let us place a thread (a piece of string) on a plane surface. The line representing the thread has no "interruptions", "cuts"; its image on the plane surface may be drawn without lifting the pointed end of a pencil. We say that the set of points forming the line is *continuous*. When the thread is cut or torn in several pieces arranged on a plane surface, the images of the lines representing the pieces of cut thread form sets of discontinuous points.

A simple formulation of the idea of continuity of points in a plane could be the following: let M be a point of the plane and one of its neighbourhoods defined by a circle of radius ε; however small, there is a neighbourhood of M of radius γ, so that $\gamma < \varepsilon$.

The image of the thread F in Fig. 3 helps us visualize the notion of order of the points in a space. No matter how the thread F is arranged, or any form obtained by its arrangement on a surface, the point M will always be situated between the points P and Q. Any point of the image of F, for instance the point N, will be "inserted", "enclosed" by points of the P and Q *categories*. But two points have a special condition: no matter how the thread is arranged, the point A will have as neighbourhood points in the Q category only, and the point B points in the category P only. The points A and B are the *marginal points* of F.

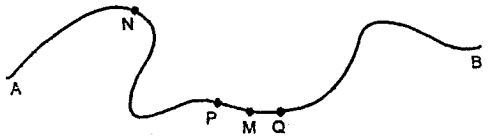


Fig. 3

In the end, in any way a segment of the thread is twisted, thrown or rotated in the air, it remains "all of a piece". Considering the thread in a Euclidean three-dimensional space as a curve, we say that the set of points forming it is "connected".

Considering the various "forms" the thread may be arranged on a surface as images through its transformation, we are led to the intuitive understanding of the fundamental properties of topological transformations: conservation through transformation of neighbourhoods, as well as of continuity, order of points and connection. This transformation is carried out without "breakings" or "junctions".

Representation in a plane of the considered transformation allows us to understand the concepts of biunivocity and bicontinuity characterising the topological transformations.

Let F be an initial form and F' its image through topological transformation (Fig. 4).

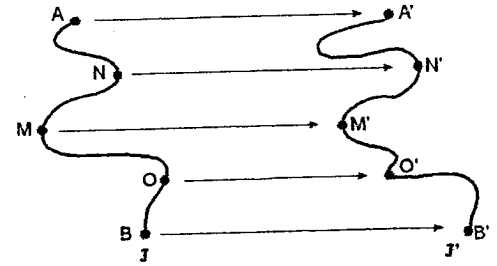


Fig. 4

Considering, for instance, the points A, N, M, O, B of the form F, in its image through transformation, there is a point, and only one, corresponding to them: A', N', M', O', B'. The transformation is *one-to-one*.

A neighbourhood of the image of the point through transformation corresponds to the neighbourhood of a point in its initial form. The transformation is *bicontinuous*.

The fact that the images of the threads F and F' in Fig. 4 have different lengths can surprise the reader. We mention again that two line segments of different lengths have the same number of points, this theorem being intuitively provable through the projection of a line segment S on a line D from a projection centre (C) (Jordan) (Fig. 5).

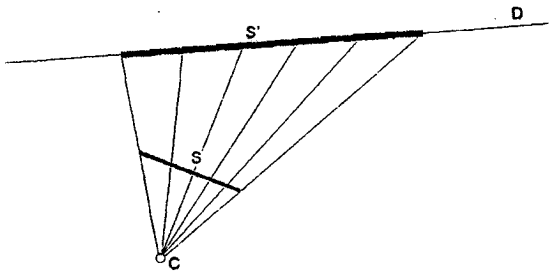


Fig. 5

A.2.2. Topological transformations and rubber membrane geometry

A very suggestive intuitive method by representing the topological transformation action (when the considered set elements are points of a plane) is that of the "rubber membrane geometry".

Let us draw on a flat rubber membrane a figure, for instance that in Fig. 6.

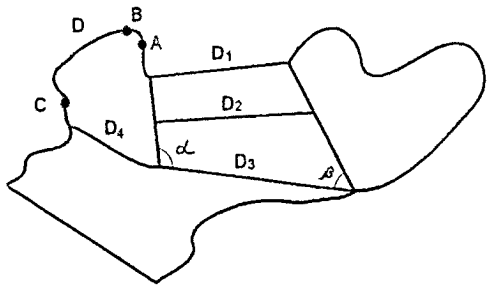


Fig. 6

By stretching this membrane in different directions, without breaking it, a new figure is obtained, for instance that on Fig. 7.

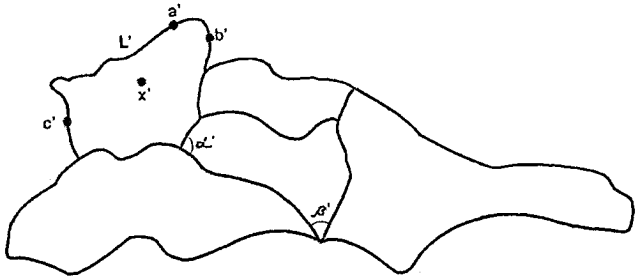


Fig. 7

By comparing the two figures, we note:

1. The closed lines remain closed lines; for instance, L has become L'.
2. The points situated on L (A, B, C) are situated on L' in the same order.
3. The point X is inside the territory defined by L; it is inside the territory defined by L'.
4. The point X in the vicinity of point B is in the vicinity of point B'.

It is intuitively proved that the neighbourhood and continuity are *preserved* through transformation and that to every point in Fig. 6 there corresponds a point, and only one, in Fig. 7.

In addition, we note the following:

1. The lines D_1, D_2, D_3, D_4 are not preserved through transformations (the straight lines no longer remain straight.
2. The parallelism is not preserved (the parallel lines in Fig. 6 are no longer parallel in Fig. 7).
3. The angle values are not preserved. For instance, α is different from α'. The line D_4 no longer forms right angles with the lines D_1 and D_2 (orthogonality is not preserved).

4. Finally, the *distance* between two points is not preserved by membrane stretching.

As a rule, the form in Fig.7 has dilated; now, if we draw a figure on the stretched rubber membrane, when we let it free from the tension points, the drawn figure will contract, the straight lines, parallelism, angle value and distances being subjected to the same transformations.

The above mentioned intuitively prove that the transformations of a topological space into another topological space or in itself preserves only the notions of point, neighbourhood, continuity and connectivity.

In Chapter 3 we will insert spaces with projective, affine, Euclidean and metric structures, which, through transformation preserve certain fundamental geometric concepts, such as straight line, parallelism, angles, distances and various other like symmetry centres, polar lines, etc.

While perusing this work, the reader will be able to note, sometimes much to his/her surprise, the role these different structures of space have in the traditional methods of musical composition.

A.3. VARIOUS STRUCTURES OF TOPOLOGICAL SPACES: PROJECTIVE, AFFINE, EUCLIDEAN AND METRIC

A.3.1. Fundamental remarks

So far, we have considered the two-dimensional space $\mathbf{R}^2$ with vectorial structure, so that each point may be associated to a couple of real numbers, its coordinates. The vectorial space is a basic structure of analytical geometry, whose domain will be further used in the process of transforming sound forms. For a mathematician, space, the plane in particular, may be imagined as having different characteristic structures, determined by their property of preserving, under the action of certain groups of transformations, some fundamental principles of geometry, such as point, line, harmonic ratio, parallelism, angle, orthogonality, circumference, distance, area, etc. The notions preserved through transformation are referred to as *invariants*. When a space has a great more invariants under the action of certain transformations, it is all the more considered to have a more profond structure. From this point of view, the topological space has the most general, the least profond structure, having as invariants the point, neighbourhood and continuity only.

In the following, we will introduce notions regarding the structures of different spaces in an intuitive way. Their rigorously mathematical definition through the agency of analytical geometry is given at the end of this chapter.

We advise the reader, especially the one less conversant with the language of analytical geometry, to skip mathematical expoundings. The parameters defining the respective transformations should be remembered in order to follow the aspect of the obtained musical image, depending on the values of these parameters. More precisely, the reader should get acquainted with the fundamental notions of the transformations, such as homology, homothety, projection centres, centre, axis and rotation angle, cross ratio and homology axis, homothety ratio, etc.

The approach to mathematical methodes may be subsequently made, chiefly when intending to apply the transformational processes to the musical creation act by means of the computer.

At the end of this work some diagrams are shown in order to facilitate the writing of computer programs capable to carry out the considered transformations, as well as for translating sounds represented in the sound space $\mathbf{R}^2$ by coordinates values, into musical notation.

A.3.2. Transformations in the general topological space

As already shown (A.1.1), the topological space has as transformation invariants *neighbourhood and continuity*; the transformation is *bicontinuous and biunique*. The transformations in Fig. 8 are topological transformations.

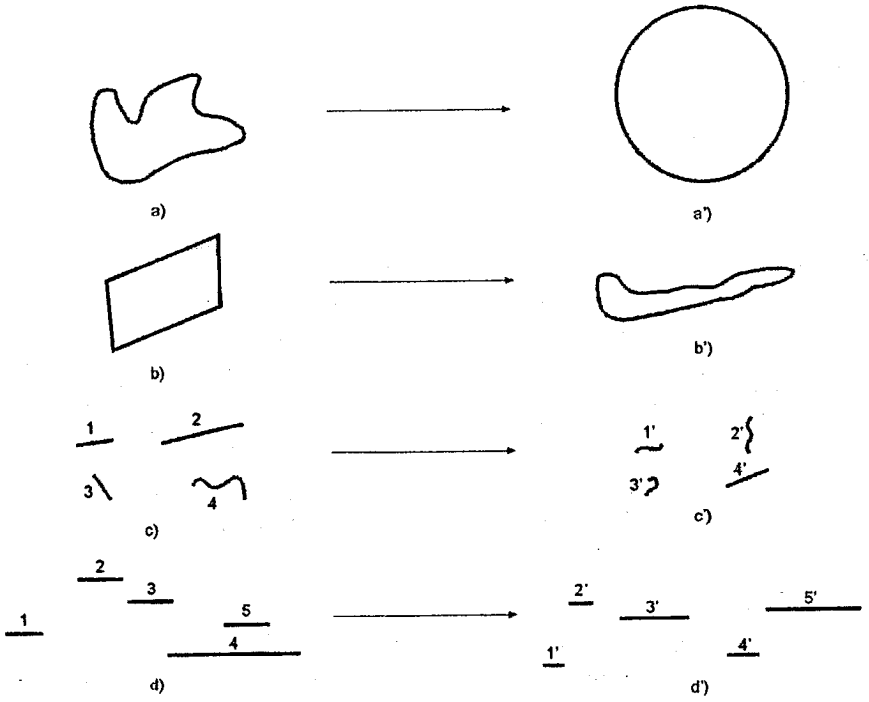


Fig. 8

A.3.3. Transformations in topological space having a projective structure

By introducing the notion of a straight line in the topological space, the mathematician determines the projective space. The projective space is the field of projective geometry. The geometry of this space is the study of the projective transformation group. Besides the topological space invariants, the projective transformations have as invariants the straight line, the harmonic ratio, the tangent, the polar line.

Intuitively, projective geometry may be understood as a ruler geometry, i.e., of the geometrical figures and geometrical transformations that can be built only with the aid of a ruler.

The straight line and the point are accepted as fundamental notions. A point P in a plane being given, the lines a, b, c are said to be concurrent in P (Fig. 9).

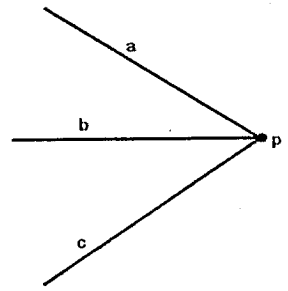


Fig. 9

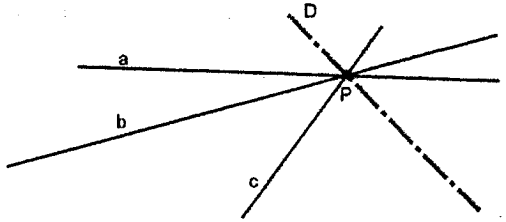


Fig. 10

A straight line D being given in a plane, we say that the straight lines a, b, c "intersect" D in the same point P (Fig. 10).

To be concurrent and to intersect in a point on a line represent reflexive, symmetrical and transitive relations.

A.3.4. Central projection

A point C referred to as projection centre, a line D and a point M situated in the plane of C and D being given, the central projection in the transformation associates to M a point M' which is on D at its intersection with the line PN referred to as projection line and crossing D and M (Fig. 11).

A set of points F being given, transformation is bijective and bicontinuous if it fulfills the following conditions:

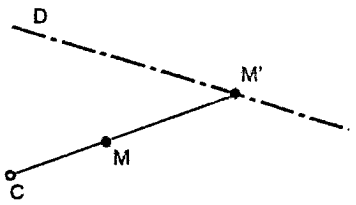


Fig. 11

1. Any projection line *PN* should intersect the set in one point only. Condition (1) is required since, when several point images of some points from *F* agglomerate in the intersection of *D* with *D'*, the bijective character of the transformation vanishes.

2. A projection line parallel to *D* should not intersect the set *F* in any point. Condition (2) is required since any point from *F* on the projection line parallel to *D* will be transformed into a point at infinity, the bicontinuous character of the transformation being thus cancelled.

When these two conditions are observed, through projection from *C* on *D*, all the points of *F* are linearized on *D* (Fig. 12).

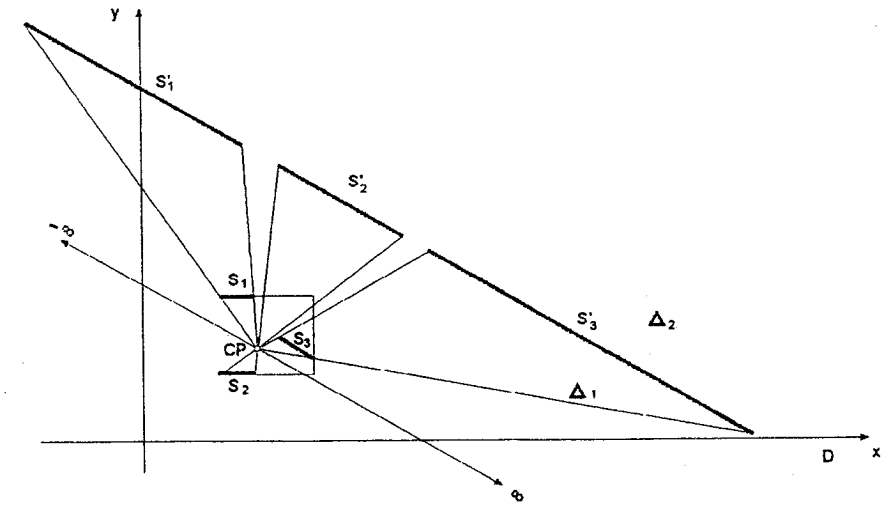


Fig. 12

A.3.5. Parallel projection

When the projection centre is at infinity, the projection is obtained through a cluster of parallel projection lines. The projection lines may make

different angles (α) with the line on which they are projected (Fig. 13). When $\alpha = 90°$, the projection is referred to as *orthogonal*.

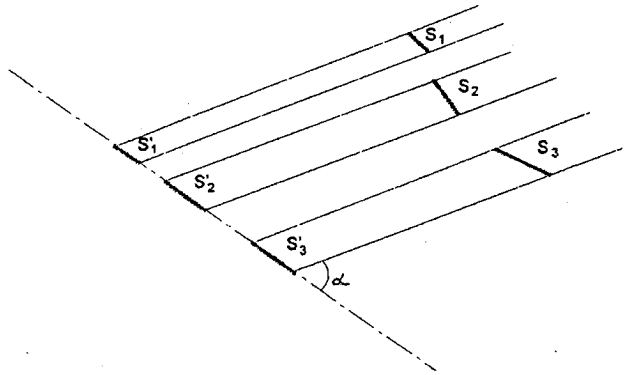


Fig. 13

A.3.6. Homology

One of the fundamental projective transformations is *homology*, i.e. the transformation of a point (or of a set of points, for instance a straight line segment or a geometrical figure) with respect to a homology axis and to a homology centre.

A point *C* (homology centre) and a straight line *a* (homology axis) being given, all the segments *AB* intersecting in the same point ***R*** on the axis *A* and included between the lines *CA* and *CB* are transformations through homology of the segment *AB* (Fig. 14).

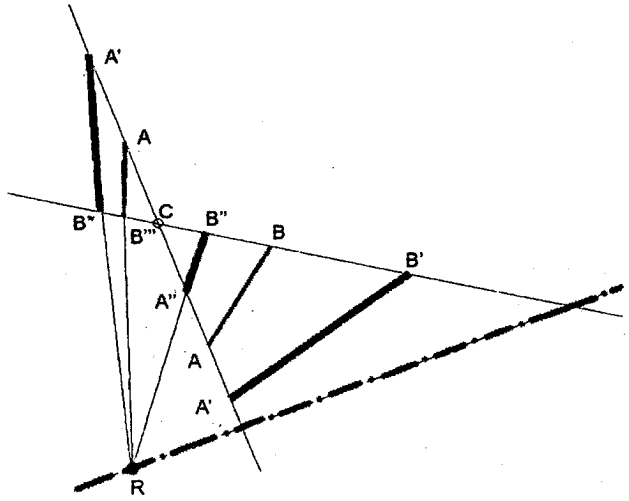


Fig. 14

The position of the points A^i (and of the points B^i, respectively, hence of the line segments A^iB^i, where ij = ', ", "', ...) with respect to the centre C and axis A, depends on the value of k biratio

$$k = \frac{\dfrac{CA}{AM_1}}{\dfrac{CA^i}{A'M_1}}$$

where M_1 is the A projection from C on A.

The following transformation of the ABC figure is a homology (Fig. 15).

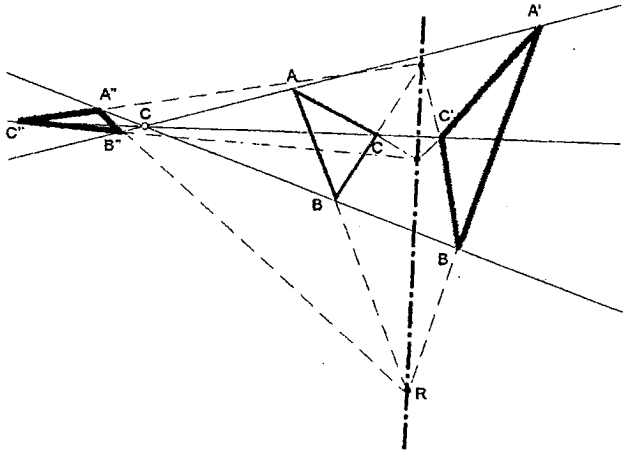


Fig. 15

AA' BB' CC' are concurrent	AC and $A'C'$)	intersect
	CB and $C'B'$)	in the same
	BA and $B'A'$)	point on a

A.3.7. Transformations in topological space having affine structure

Assigning to the geometrical space the concept of parallelism, we obtain the space with affine structure, the affine space. The geometry of this space is the study of the affine transformation group. Intuitively, the affine geometry may be understood as the ruler and translator geometry, the figures and transformations in the affine geometry being possibly carried out by means of the ruler and of the translator. Invariants: projective invariants plus parallelism, cross section ratios, middle, centre of symmetry,

asymptote, etc. Fundamental affine transformations: parallel homology, translation, homothety.

A.3.8. Parallel homology

The parallel homology is a homology whose centre is at infinity, illustrated in Fig. 16.

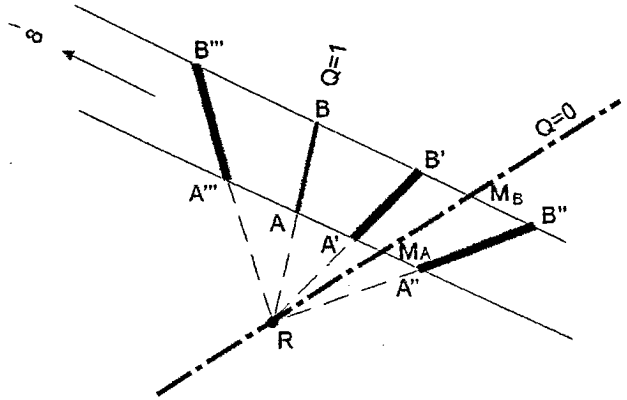


Fig. 16

From this point of view, the parallel homology belongs both to the projective space and to the affine one; we could say that it makes the connection between the projective geometry and the affine one. The following image illustrates the transformation of the A B C figure through parallel homology (Fig. 17):

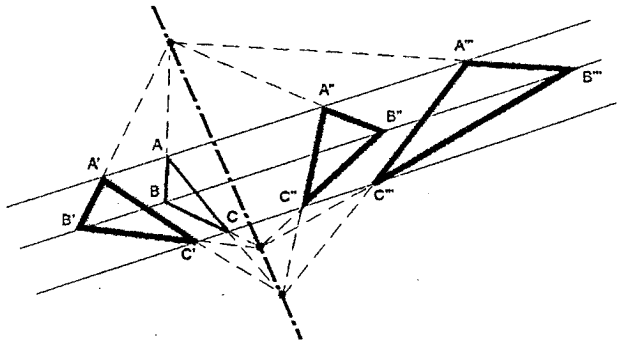


Fig. 17

A A' BB' CC' are colinear	A B and $A'B'$)	intersect
	B C and $B'C'$)	on the
	C A and $C'A'$)	same line

The position of the transformed image depends on the value of the k ratio of the points A, A^i M (B, B^i, C, C^i, respectively), where: $i, j, l = 1,"...,^{n\times1}$

$k = \frac{AM}{A^iM}$ and on the value of angle α of the projective line with axis Q. In Fig. 17 $\alpha = 90°$, hence the projection is orthogonal.

A.3.9. Translation

Translation is a geometrical transformation, defined by a vector V referred to as *director vector,* so that the point P with $PP' = V$ corresponds to the point P'. The translation maintains the distances between points, their collinearity and the line parallelism, transforming a figure into another one equal and parallel to the given one. A translation is illustrated in Fig. 18.

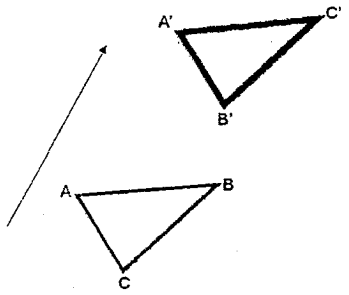


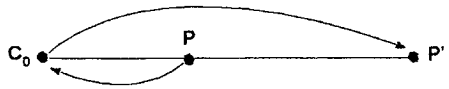
Fig. 18

AA', BB', CC' are parallel
AB and *A'B*, *BC* and *B'C'*, *CA* and *C'A'* are parallel.

A.3.10. Homothety

Homothety represents the transformation of a point related to a homothety center C (Fig. 19).

Fig. 19



The points C, P and P' are collinear. The position of point P' relative to P and C depends on the value of ration q

$$q = \frac{m}{b}$$

A transformation through homothety is represented in Fig. 20.

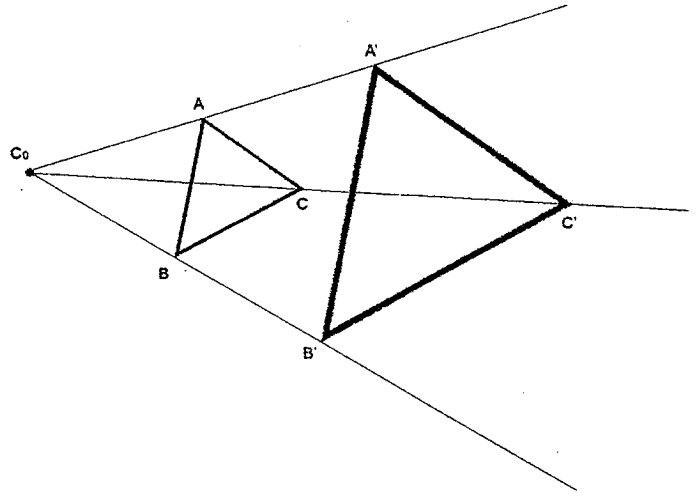


Fig. 20

AA', BB', CC' are concurrent in C_0.
AB and *A'B'*, *AC* and *A'C''*, *BC* and *B'C'* are parallel.

A.3.11. Transformations in topological space having Euclidean structure

The Euclidean space geometry is characterized by the property of preserving while transforming a geometrical figure all the affine invariants, plus the angles (particularly the orthogonality), circumference, equilaterality, symmetry axis, etc. When carrying out the transformations of the Euclidean space geometry, beside the tools used in the affine space, there is also another tool: the calliper. The basic transformation is *rotation.*

A.3.12. Rotation

Rotation is a transformation defined by a point C (rotation center) and parameter α (angle of rotation), transformation which associates to a point P the point P', so that $OP = OP'$ and the angle $POP' = \alpha$. A straight line is

transformed through rotation into another straight line, so that their angle is equal to the angle of rotation. Angles are kept invariant through rotation (Fig. 21).

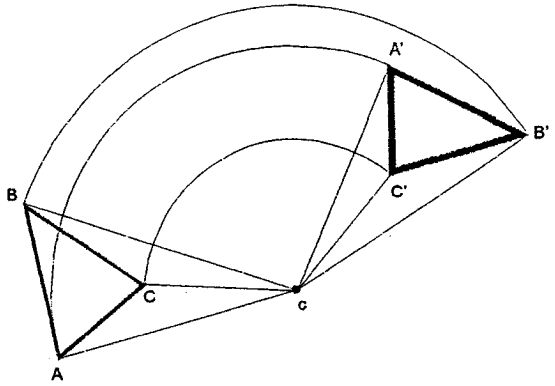


Fig. 21

A.3.13. Isogonal transformations. Inversion

The isogonal transformations leave invariant the angles of curves. Out of the group of isogonal transformations we will examine *inversion.*

Inversion is a transformation by which a point A is associated to another point A', so that the product of the distances from the point A to another point O, pole of inversion, and from the point A' to the same point O, has a given value K. The module k has an area dimension. The points O, A and A' are collinear

$$OA * OA' = k$$

The inversion module may be positive or negative. For the sign +, the point A and its transform A' are on the same side of the pole O; for the sign –, the pole O separates the point A from its transform A' (Fig. 22).

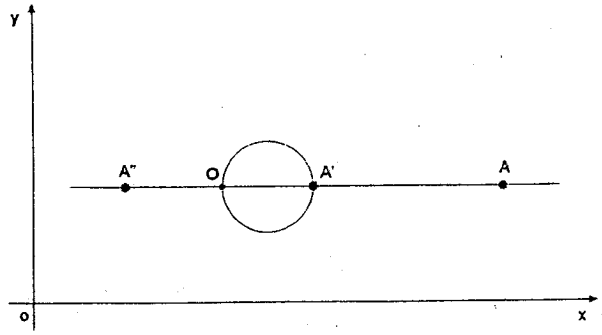


Fig. 22

The circle with its centre in the pole O and radius K is referred to as circle of inversion.

A.3.14 Transformations in topological space having a metric structure

The concept of distance between two points defines the metric space and the geometry associated with it, i.e., the metric geometry, characterized by the property of preserving, when transforming figures, all Euclidean invariants, plus the distances, the surfaces and the volumes.

In the process of transforming figures through rotation, to the preceding tools, the marked off in units of measure ruler is added (Fig. 23).

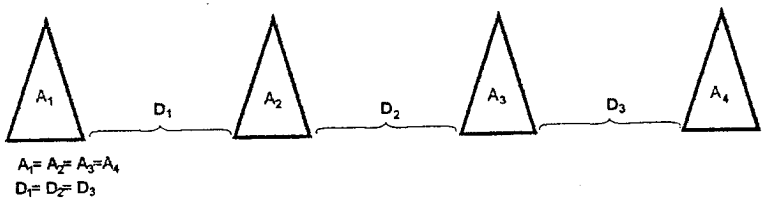


Fig. 23

$$A_1 = A_2 = A_3 = A_4$$
$$D_1 = D_2 = D_3$$

Structures and transformations of the three-dimensional $\mathbf{R}^3$ space

The various structures wich characterize the $\mathbf{R}^2$ space can be induced in the three-dimensional $\mathbf{R}^3$ space.

A.3.15. Transformations in the topological space $\mathbf{R}^3$ endowed with projective structure

Projection in $\mathbf{R}^3$

Projection in $\mathbf{R}^3$ is the geometrical process of representing a figure of the $\mathbf{R}^3$ space on a plane (particularly on the plane defined by the Ox, Oy axes) as the interaction of the plane with the projection lines of all the points of the considered figure. *Conical projection* represents the projection made with the projection lines which pass through a steady point (projection center).

Fig. 24 represents a mapping of a figure F through central projection from the centre PC on the plane P'.

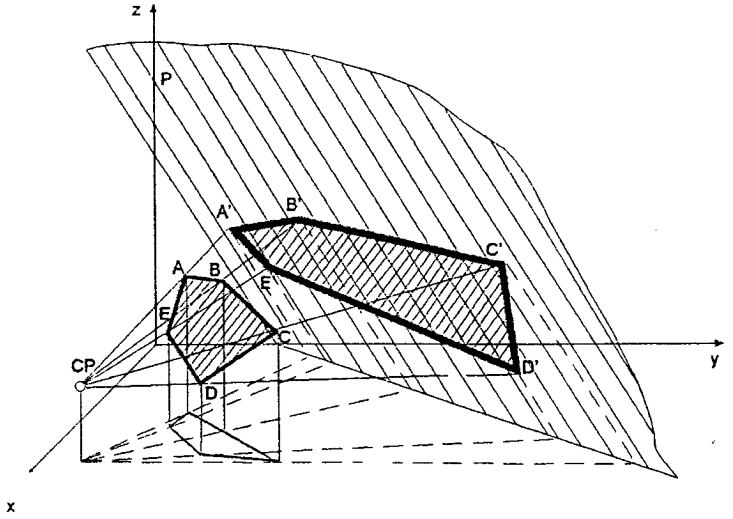


Fig. 24

A particular case of interest for this work is the projection of a plane figure from $\mathbf{R}^3$ on the plane formed by the Ox and Oy axes in the space $\mathbf{R}^2$. Fig. 25 illustrates such a mapping through oblique cylindrical projection.

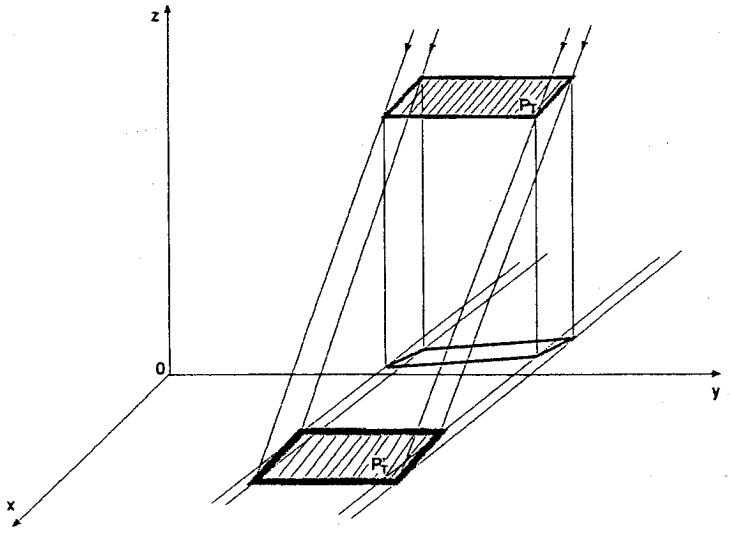


Fig. 25

The following three cases are noteworthy: the image obtained through projection no longer preserves the topological characteristics of the initial figure:

1. Cp is situated relatively to F so that part of the points of F, $\{M_1\}$, are "over" it, and the other part $\{M_2\}$ "below" it

$$z(M_1) > z(Cp) > z(M_2)$$

The transformation is not continuous, hence it is not topological. The points of F are discontinuously scattered in $\mathbf{R}^2$, except a point m which, being on a projecting line parallel to Ox is sent to infinity (Fig. 26).

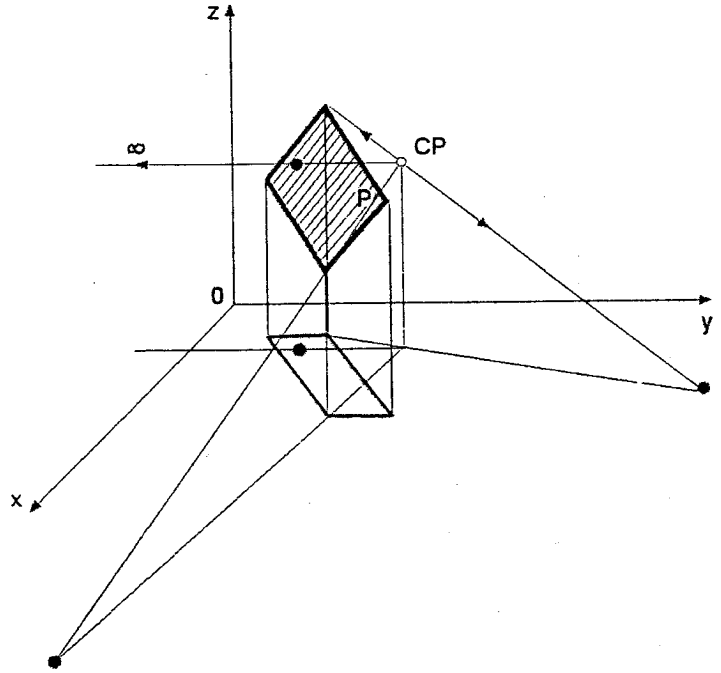


Fig. 26

2. Cp is situated in the same plane P with F. Through projection, all the points $\{M'\}$ of the transformed figure F "settle" on the straight line D representing the intersection of P with $\mathbf{R}^2$. The transform loses its bijectivity, hence the topological character (A.1.1) (Fig. 27).

3. Cylindrical projection: projection lines parallel to space $\mathbf{R}^2$. The image through projection of F is sent to infinity: we have a null transformation, $\alpha = 0; 180°$ (Fig. 28).

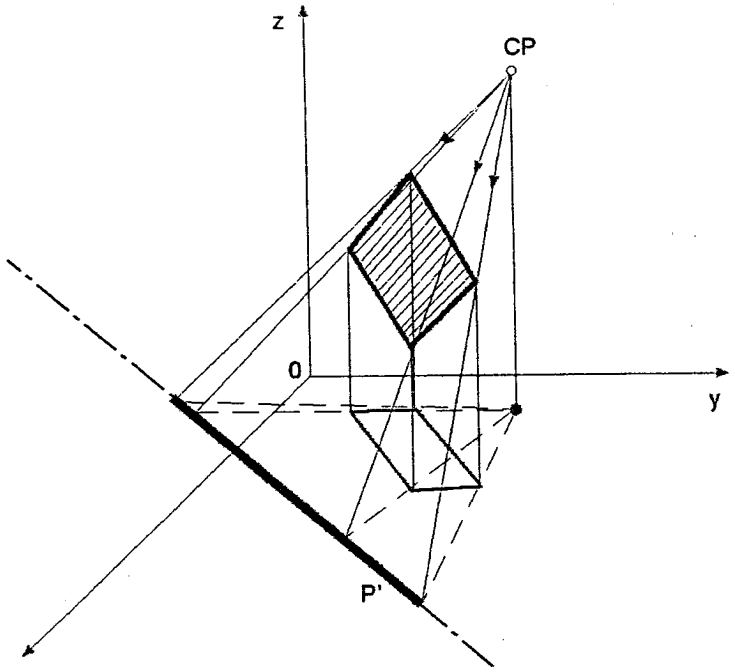


Fig. 27

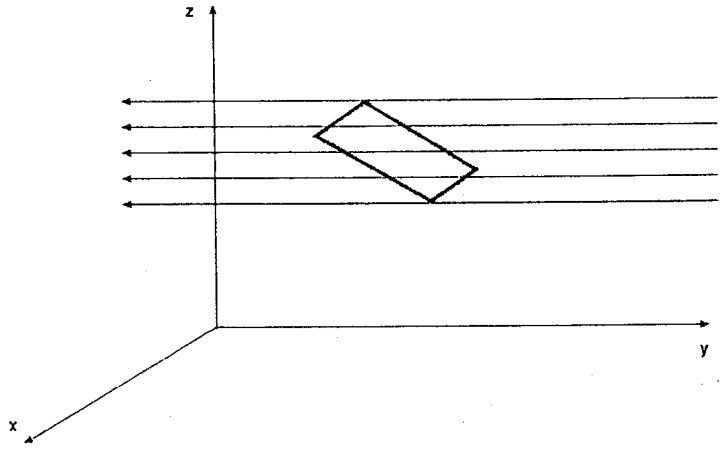


Fig. 28

A.3.16. Homology in R^3

The transformations through central and parallel homology cannot take place, by definition, otherwise than in a plane. As a consequence, the homology mappings in $\mathbf{R}^3$ amount to plane mappings from $\mathbf{R}^3$; the study of figure transformations through homology in $\mathbf{R}^2$ may be extended to transformations of this type from any plane in $\mathbf{R}^3$.

A.3.17. Transformations in the topological space R^3 having affine structure

Translation in $\mathbf{R}^3$ of a plane figure may be made in terms of any trajectory in this space. As for homothety, it can be related to centres included in the same plane as the figure or to centres not included in the figure plane. We will consider in this work only the homothety with centres in the same plane with the figure. The study of transformations through homothety in $\mathbf{R}^2$ can be extended to transformations through homothety in any plane of $\mathbf{R}^3$.

A.3.18. Transformations in the topological space R^3 having Euclidean structure. Rotation in R^3

A plane figure in $\mathbf{R}^3$ being a set of coplanar points in $\mathbf{R}^3$ its rotation amounts to the rotation of part of a plane in $\mathbf{R}^3$ round an axis (Fig. 29).

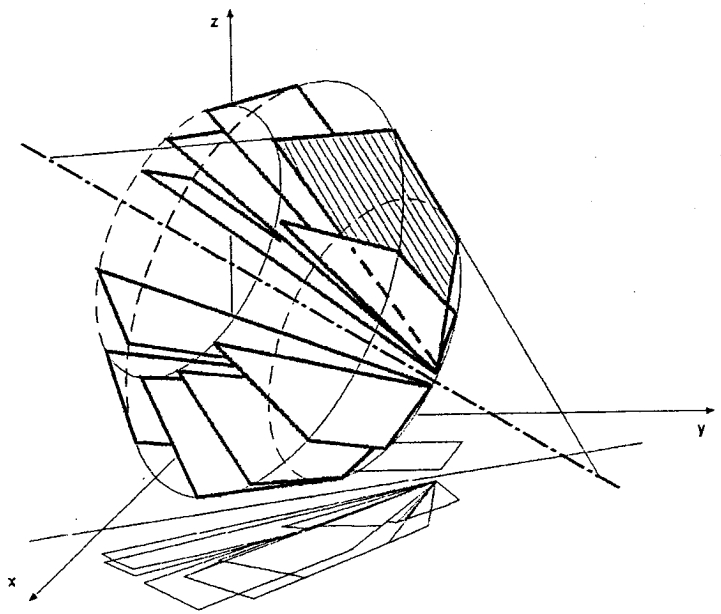


Fig. 29

A.3.19. Rotation round an axis included in the same plane with the figure and intersecting it (Fig. 30)

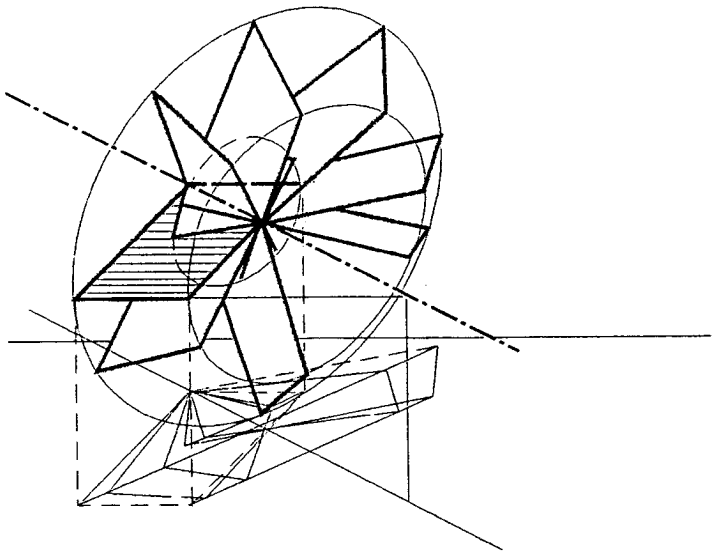


Fig. 30

A.3.20. Compound transformations in the topological space R^3

Transformations in $\mathbf{R}^3$ may be compound. In the further examples we will study a product of translation, rotation and projection transformations in $\mathbf{R}^3$, as being of interest for this work.

(a) The figure F represented in $\mathbf{R}^2$ is "lifted" through translation in $\mathbf{R}^3$ with respect to a vector perpendicular to $\mathbf{R}^2$ (F'). The "height" of F is expressed by the value of the z coordinates, $z > 0$ (Fig. 31).

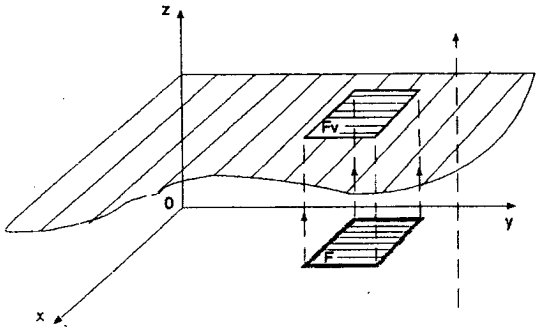


Fig. 31

(b) A rotation axis is defined in the plane Q, parallel to $\mathbf{R}^2$. F' is also situated in Q. ($z(Q) = z(F)$). D is defined by a point P of coordinates x, y, z, P being by definition equal to $z(Q)$) and by an angle α made by D with the axis Ox (Fig. 32).

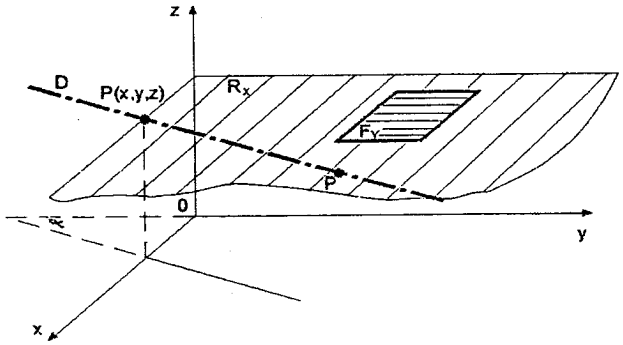


Fig. 32

(c) F' is rotated around D at an angle β (Fig. 33).

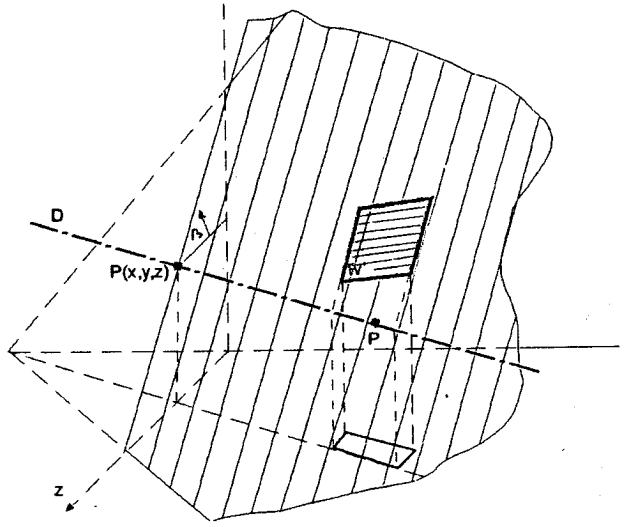


Fig. 33

(d) F is projected from a projection centre or through cylindrical projection in the plane defined by the Ox and Oy axes (Figs. 34a, 34b).

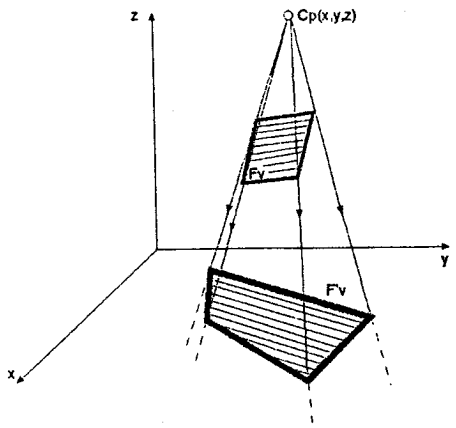


Fig. 34a

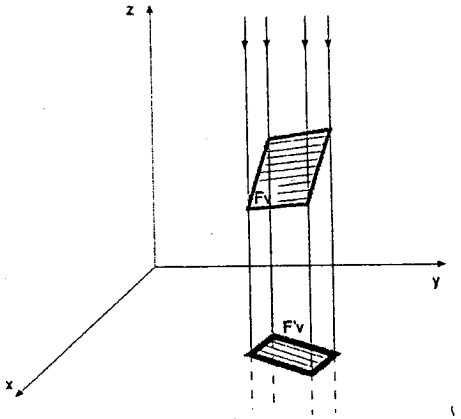


Fig. 34b

A.4. ANALYTICAL ASPECTS OF SOME FOREGOING CONSIDERED TRANSFORMATIONS

A.4.1. Projective transformations in R^2

The projective structure of the plane $\mathbf{R}^2$ is the ensemble of plane $\mathbf{R}^2$ invariant properties under the action of the projective group transformations, analytically stated by the formulae of the $(x, y) \rightarrow (x', y')$ type

$$x' = \frac{ax + bx + c}{px + qy + r}, \quad y' = \frac{dx + ey + f}{px + qy + r}, \quad \begin{vmatrix} a & b & c \\ d & e & f \\ p & q & r \end{vmatrix} \neq 0$$

Considering that certain points, more exactly the points situated on the straight line of equation

$$px + qy + r = 0$$

have as images, through written transformation, points with infinite coordinates, but with a generally finite y' / x' ratio, it is proper to complete the plane $\mathbf{R}^2$ with inadequate points, all of them situated on a straight line, referred to as the straight line of infinity, denoted by d ∞ and which, along with $\mathbf{R}^2$, form the real projective plane $P^2(R) = R^2 \cup$ d ∞. The projective transformations plot lines in lines and leave invariant the biratio of four collinear points, the biratio being defined by

$$(ABCD) = \frac{\overline{AC}}{\overline{BC}} : \frac{\overline{AD}}{\overline{BD}}$$

A.4.2. Homology

Homology is a projective transformation given by equations such as:

$$x' - x_0 = \frac{x - x_0}{1 + \lambda l(x, y)}, \quad y' - y_0 = \frac{y - y_0}{1 + \lambda l(x, y)}$$

where $l(x, y) = ax + by + c$.

The equation $l(x, y) = 0$ defines a straight line referred to as *homology axis*. The point C of coordinates (x_0, y_0) is called *center of homology*. When C is not situated on the homology axis, homology is referred to as non-special and the straight lines passing through C are invariant, since both C and the cross points of such lines with the axis of homology are fixed points.

Biratio of the points C, M_1, M, M', $\quad Q = \dfrac{\dfrac{CM}{MM_1}}{\dfrac{CM'}{M'M_1}}$

where M_1 is the cross point of the line (C, M) with the homology axis, and M' is the transform of the point M, arbitrarily selected in the plane. This is illustrated in Fig. 35.

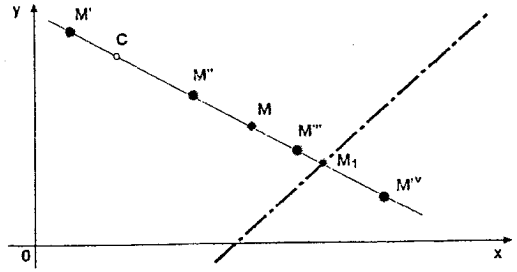


Fig. 35

A.4.3. Three relative positions of the homology axis, the homology center and the point to be transformed

The position of point M with respect to the homology axis and the homology center may exhibit three cases (Fig. 36a,b,c):

(I) M between center and axis

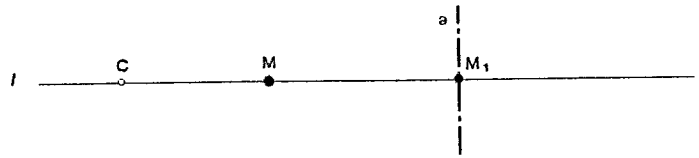


Fig. 36a

(II) Axis between center and M

Fig. 36b

(III) Center between M and axis

Fig. 36c

A.4.3a: *The homology biratio k cannot take continuous values from $-\infty$ to $+\infty$*

There are certain critical values which given to k, would send the point M' to infinity. More exactly, this situation occurs when, for the cases I, I, III, the value of k is equal to

$$\frac{\alpha}{1-\alpha}, \quad \frac{\alpha}{\alpha-1} \quad \text{and} \quad \frac{\alpha}{\alpha+1}$$

where $\alpha = \dfrac{CM^2}{CM_1}$

A.4.4. Relative position of the transformed image with respect to the value of the homology biratio

The following diagram indicates the place of point M' in the various zones of the line CM, depending on the value of biratio K (Fig. 37).

[2] Indeed, when the point M' is sent to infinity, the distances CM' and M_1M' in the biratio:

$$k = \frac{\dfrac{CM}{M_1M}}{\dfrac{CM}{M_1M'}}$$

tend to infinity, and the ratio $\dfrac{CM'}{M_1M'}$ tends to 1. Consequently, k should avoid taking the value $\dfrac{CM}{M_1M}$. By dividing $\dfrac{CM}{M_1}$ by CM_1 and substituting M_1M in case I with $CM_1 - CM$, in case II with $CM - CM_1$, in case III with $CM + CM_1$ we obtain:

in case I $$\frac{CM}{M_1M} = \frac{\dfrac{CM}{CM_1}}{\dfrac{M_1M}{CM_1}} = \frac{\dfrac{CM}{CM_1}}{\dfrac{CM_1 - CM}{CM_1}} = \frac{\dfrac{CM}{CM_1}}{\dfrac{CM_1}{CM_1} - \dfrac{CM_1}{CM_1}} = \frac{\alpha}{1-\alpha}$$

in case II $$\frac{CM}{M_1M} = \frac{\dfrac{CM}{CM_1}}{\dfrac{CM}{CM_1} - \dfrac{CM_1}{CM_1}} = \frac{\alpha}{\alpha-1}$$

in case III $$\frac{CM}{M_1M} = \frac{\dfrac{CM}{CM_1}}{\dfrac{CM}{CM_1} + \dfrac{CM_1}{CM_1}} = \frac{\alpha}{\alpha+1}$$

Case I – M situated between C and M_1
Case II – axis M_1 between C and M
Case III – C between M and M_1

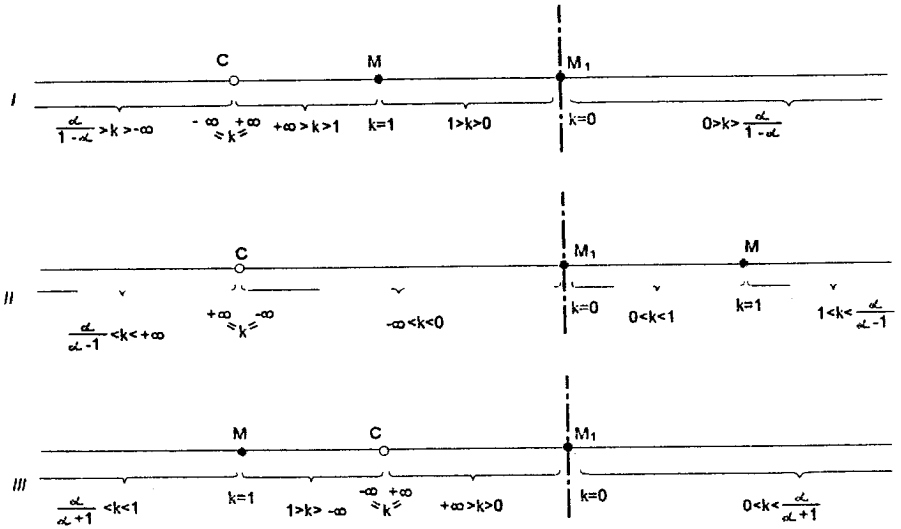


Fig. 37

Note that for $k = 1$, M' coincides with M (M is transformed into itself. For $k = 0$, M' coincides with M_1 for $k = \pm\infty$, M coincides with C. When M is on the axis or in the homology centre, M' coincides with M, M_1 or C, respectively, for any value of k.

A.4.5. Graphical representation of the critical values of the homology biratio

Fig. 38 graphically illustrates the fact that, for a certain critical value of the biratio k

$$k \in \frac{\alpha}{1-\alpha}, \frac{\alpha}{\alpha-1}, \frac{\alpha}{\alpha+1} \qquad \text{(A.3.3)}$$

a certain point belonging to a connected set of points of finite coordinates – in this case a point belonging to the straight line D – is transformed through homology in a point of infinite coordinates. Fig. 38 illustrates case I (M_i) ($i = a, b, \ldots e$) situated between C and a (A.3.4). The point M_a being situated on the axis will be transformed into itself for any value of k.

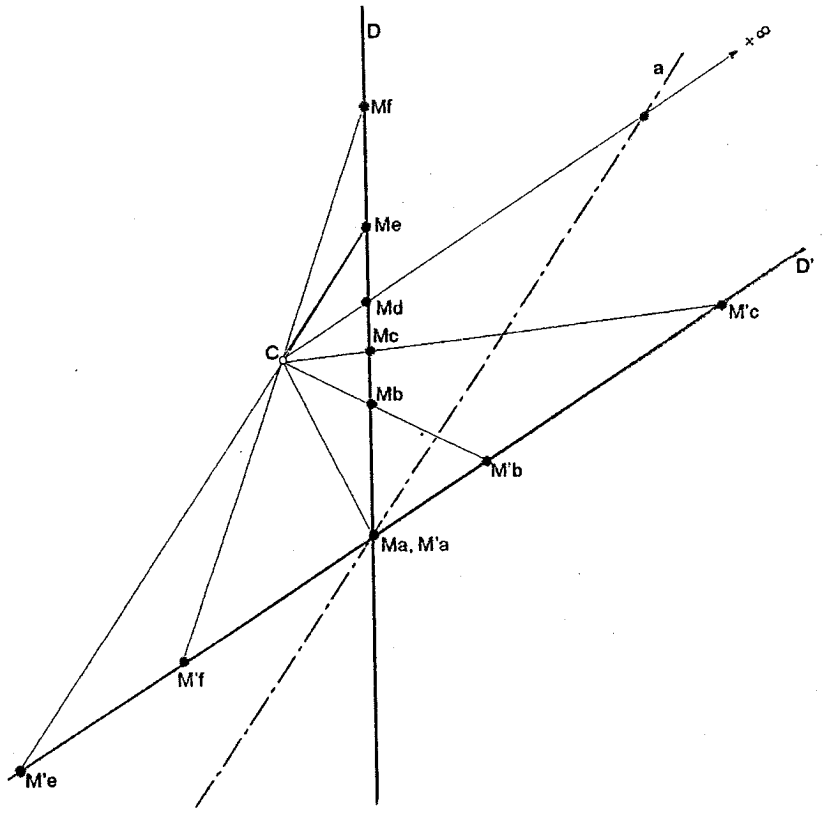


Fig. 38

When $k \to \frac{\alpha}{1-\alpha}$, the points M_b, M_c next to the point M_d will tend, through transformation on D', to $+\infty$ (M'_b, M'_c).

For $k = \frac{\alpha}{\alpha-1}$, M'_d the transform of M_d will "vanish" at $+\infty$ since the straight line CMd is *parallel* to D'.

For $\frac{\alpha}{\alpha-1} > k$, the points next to M_d (M_c, M_f) will "appear" on D' from $-\infty$.

The connected set of points D loses its continuity through transformation, compromising the topological aspect of the transformation (Fig. 38).

A.4.6. Maintaining the continuity of a straight line segment

When transformation by central homology is applied to a straight line segment, in order to preserve the continuity of its image by transformation, it is necessary to make sure that none of its points is sent to infinity. For instance, in Fig. 38 the intervals (M_a, M_e), (M_b, M_c), (M_a, M_g), (M_e, M_f) belonging to D are transformed into the intervals (M'_a, M'_e), (M'_b, M'_c), (M'_a, M'_g), (M'_e, M'_f), respectively, belonging to D'. The interval (M_c, M_d) cannot be transformed into a continuous interval in D', as it includes the point M'_d which is sent to infinity.

Thus, in order to preserve the topological aspect of the transformation k should not take equal values to

$$\frac{\alpha}{1-\alpha}, \quad \frac{\alpha}{\alpha-1}, \quad \frac{\alpha}{\alpha+1}$$

respectively, for any point of the segment. Otherwise, near the critical value, the segment would "break" through transformation, its ends tending towards $+\infty$ and $-\infty$ and compromising the topological character of the transformation. In order to be sure that a straight line segment which has to be transformed through homology preserves its continuity, it is necessary and sufficient to make sure that the points at the segment's ends are not sent to infinity for the k value selected for transformation.[3]

[3] **Theorem.** In homology mapping, in order not to geopardize the topological character of transformation, when the set of points is a straight line segment not intersecting the homology axis, the biratio k should not take values included in a finite interval, and when the segment intersects the homology axis, k should not take values included in the union of two infinite intervals.

The first result shows that α is always an increasing or decreasing function when the relevant point M describes a segment.

Proof. Let us study the variation of α when M covers a segment S (Fig. 39).

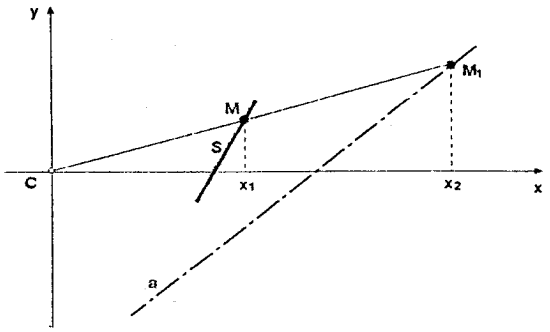


Fig. 39

Let us provisionally use a coordinate system with its origin in the homology center:
Let us consider:

the equation of radius (CM_1) $\quad y = mx$ (1)
the segment equation (CM) $\quad y = m_1x + b_1$ (2)
the equation of homology axis (A) $\quad y = m_2x + b_2$ (3)

$$\alpha = \frac{CM}{CM_1} = \frac{X_1}{X_2} \qquad \text{(A.4.3a)} \qquad \alpha = \frac{CM}{CM_1} = \frac{X_1}{X_2} \tag{4}$$

By substituting y from (1) in (2) and (3), we obtain:

for X_1 $\quad my_1 = m_1y_1 + b_1 \rightarrow X_1 = \dfrac{b_1}{m - m_1}$

for X_2 $\quad my_2 = m_2y_2 + b_2 \rightarrow X_2 = \dfrac{b_2}{m - m_2}$

Substituting in (4) we obtain:

$$\alpha \quad \frac{b_1}{b_2} \times \frac{m - m_2}{m - m_1}$$

We find the derivative of α depending on m (since m varies while traversing S)

$$\frac{d}{dm} = \frac{b_1}{b_2} = \frac{(m - m_1) - (m - m_2)}{(m - m_1)^2} = \frac{b_1}{b_2} = \frac{m_2 - m_1}{(m - m_1)^2}$$

Since the denomination is a quadratic expression, the sign of $\dfrac{d\alpha}{dm}$ is always the same as the sign of $m_2 - m_1$ expression. Consequently, the value of α is always either increasing or decreasing during the S travel, depending on the fact that $m_2 > m_1$ or $m_2 < m_1$.

As a conclusion, the *minimum* α and *maximum* α values are situated at the end of the segment S.

The second result shows that $\dfrac{\alpha}{\alpha - 1}$ is a decreasing function of α in the interval $(-\infty, 1)$ and $(1, +\infty)$.

Now, the case II is studied (Fig. 40).

Let α min be the minimum of α at the segment ends, and α max the maximum of α at the segments ends.

$$k \text{ min.} = \frac{\alpha \text{ min}}{\alpha \text{ min} - 1}; \qquad k \text{ max.} = \frac{\alpha \text{ max}}{\alpha \text{ max} - 1}$$

Two examples of max and min are illustrated in Fig. 41.

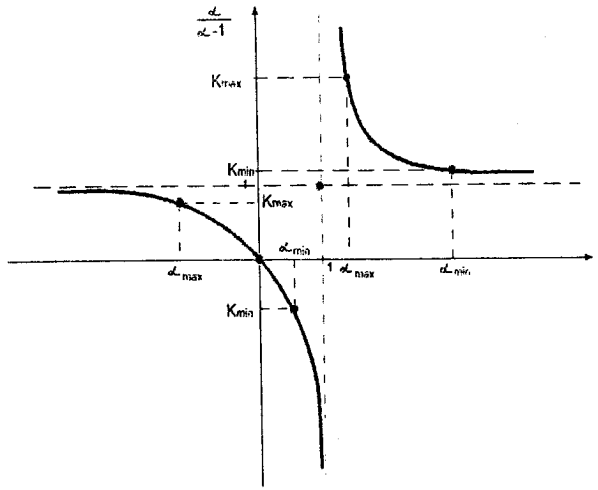


Fig. 40

In case a segment intersects the homology axis, we have a situation as in Fig. 41.

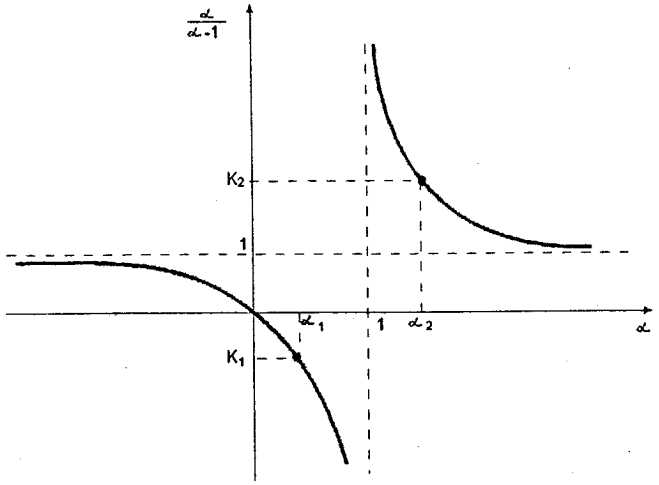


Fig. 41

k cannot take values generating y min and y max. k may take the following values:

$$k > \frac{\alpha \max}{\alpha \max - 1} \quad \text{or} \quad k < \frac{\alpha \min}{\alpha \min - 1}$$

k_1 and k_2 are the values of k corresponding to segment ends, $k = D$ corresponding to the point of intersection with the homology axis. Here, k cannot take values smaller than k_1 and greater than k_2 ($k_1 < k < k_2$).

If one aims to transform a finite set of segments, k cannot take the values:

$$\frac{\alpha}{1-\alpha}, \quad \frac{\alpha}{\alpha-1}, \quad \frac{\alpha}{\alpha+1}$$

respectively, for any one of the points forming the given segments. In other words, if the points p, q, r belong to some given segments, the value of k should be different from the critical values:

$\left(\frac{\alpha}{1-\alpha}\right.$, etc.) corresponding to the points p, q, r.

In case one aims to transform a great set of segments or non-collinear points, it seems more advantageous to avoid the critical values of k for all the points included in the rectangle defined by the convex envelope passing through the coordinate points x, y of the considered sets of minimum and maximum value, respectively (Fig. 42).

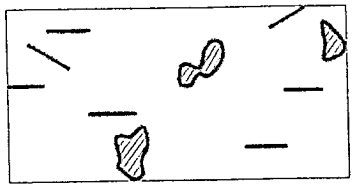


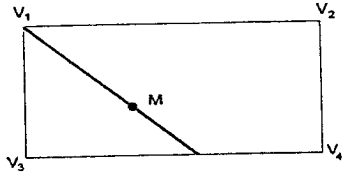
Fig. 42

It may be easy to demonstrate that it is enough to consider the values of the ratios

$$\frac{\alpha}{1-\alpha}, \quad \frac{\alpha}{\alpha-1} \quad \text{and} \quad \frac{\alpha}{\alpha+1}$$

respectively, corresponding to the corners of the convex envelope (Fig. 43)

Fig. 43



and the above results show that the corresponding critical values of an arbitrary point M in the rectangle is included between the smallest and the greatest of the critical values corresponding to the corners V_1, V_2, V_3, V_4 of the rectangle. To conclude, for instance, for the case II

$$k \min = \frac{\alpha \min}{\alpha \min - 1} \text{ and } k \max = \frac{\alpha \max}{\alpha \max - 1}$$ are the minimum and maximum values, respectively, of the expression $\frac{\alpha}{\alpha - 1}$ of the four angles, when the admissible values of k are $k < k \max$, or $k > k \min$.

A.4.7. Particular case of a line segment transformation

We have shown (A.3.6) that a homology center C and a homology axis A being given, all the segments of the straight lines intersecting A in the same point and included between CA and CB are homologous. Particularly, a segment parallel to the homology axis is transformed into segments meeting the axis to infinity (Fig. 44).

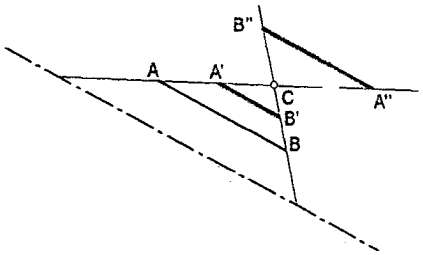


Fig. 44

A.4.8. Behaviour of a vector under the action of homology mapping

In (A.4.3) we have studied the variation in position of a point transformed through homology, depending on the relative position of the axis, of the centre and of the value of the homology biratio. For various homologies with the same axis, the same centre C, and different values of the biratio k, the transform of point M coincides with M for $k = 1$, with the intersection point M_1 of the line CM with the axis for $k = 0$ and with the point C for $K = \pm\infty$. We will examine now the behaviour of a vector V in the space $\mathbf{R}^2$ for the three specified cases (A.4.4) (Figs. 45a, 45b, 45c).

Case I

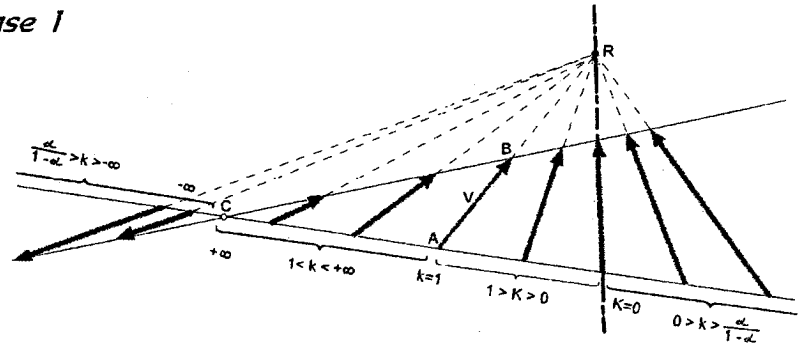


Fig. 45a

In a similar way, the expression relevant for the ratios $\frac{\alpha}{\alpha-1}$ and $\frac{\alpha}{\alpha+1}$ may be deduced.

Case II

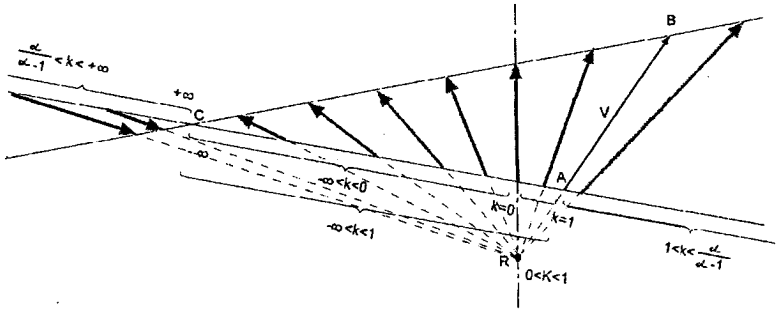


Fig. 45b

Case III

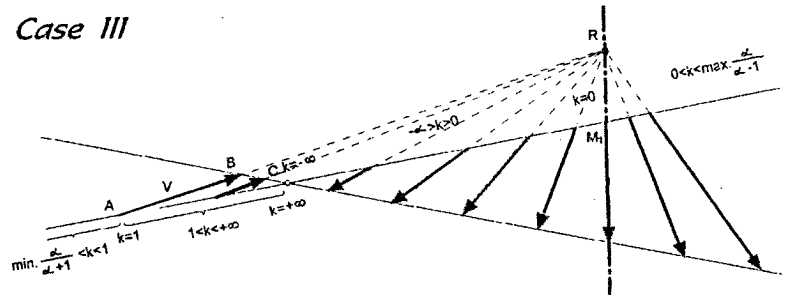


Fig. 45c

We may synthesize the following general remarks:

For $k = 1$, V is transformed into itself; for $k = 0$, V is included in the line representing the homology axis; for $k = \pm\infty$, V agglomerates (aggregates) itself in the point represented by the homology centre C.

For $k = G_i$ $\left(G_i = \frac{\alpha}{1-\alpha}, \frac{\alpha}{\alpha-1}, \frac{\alpha}{\alpha+1}\right)$,

one or some of the vector points is/are sent to infinity, hence, the vector loses its connectivity (the function representing it loses its continuity) and the vector "breaks".

A.4.9. Study of the three cases of relative position with respect to homology axis and center

Case I

While k covers the interval $(1, +\infty)$, the vector "compresses", tending to agglomerate in the point center of homology C; while k covers the interval

$\alpha \min$

$(1, \dfrac{\alpha \min}{1 - \alpha \min})$, the vector dilates;

While k covers the interval $(\dfrac{\alpha \max}{1 - \alpha \max}, -\infty)$, V "compresses" tending to agglomerate in C.

Case II

While k covers the interval $(1, -\infty)$, V "compresses" tending towards the point C.

For k: $(1, \dfrac{\alpha \min}{\alpha \min - 1})$, V dilates; for k $(\dfrac{\alpha \max}{\alpha \max - 1}, +\infty)$

V "compresses" tending to C.

Case III

While k covers the interval $(1, \dfrac{\alpha \max}{\alpha \max + 1})$ V "dilates";

For k: $1 \rightarrow +\infty$, V "compresses".

While k covers the interval $(\min \dfrac{\alpha}{\alpha + 1}, -\infty)$

V is compressed tending to C (Fig. 46).

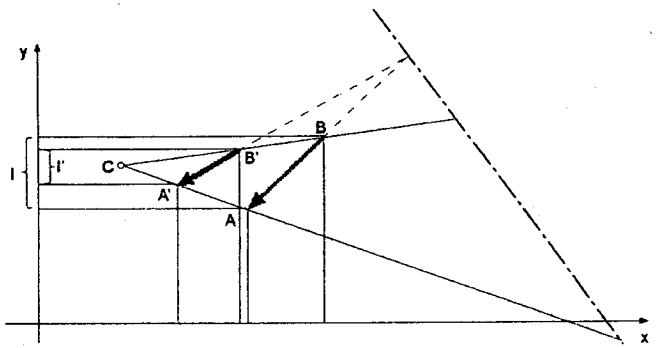


Fig. 46

By compression and dilatation we mean the increase and decrease of vector V length, hence of the value of its components represented by the projection of the vector on the coordinate axes.

Remark. When the biratio k changes its sign, the vector V changes of sign (in the neighbourhood of the centre) and of direction (in the neighbourhood of the axis).

A.4.10. Behaviour of a geometric figure under the action of transformation through homology

Figs. 47a, b, c illustrate the behaviour of the geometric figure F, the triangle ABC, in the three cases of positions realtive to the figure, the centre and the axis of homology.

As in the case of a vector behaviour, we may write the following remarks:

For $k = 1$, F is transformed into itself; for $k = 0$, the entire figure is included, "flattened", on the axis of homology. The fact should be noted that, in this case, the mapping is no longer topological, since the condition of biuniqueness is not observed, a single point in the image obtained through homology corresponds to differents points in the "source" figure.

For $k = \pm\infty$, F is agglomerates in the centre of homology, theoretically reduced to a triangle of infinitely small dimensions. For certain critical values of k

$$\left(\frac{\alpha}{1-\alpha}, \frac{\alpha}{\alpha-1}, \frac{\alpha}{\alpha+1}\right),$$

certain points of F may be sent to infinity, hence the transformation is no longer continuous, the figure loses its connectivity.

Case I

While k covers the interval $(1, +\infty)$, the figure F "compresses", tending to be overlap the centre of homology C (Fig. 47a).

While k covers the interval $(1, \dfrac{\alpha}{1 - \alpha})$ F is dilated.

While k covers the interval $\left(\frac{\alpha}{1-\alpha}, -\infty\right)$ F "compresses".

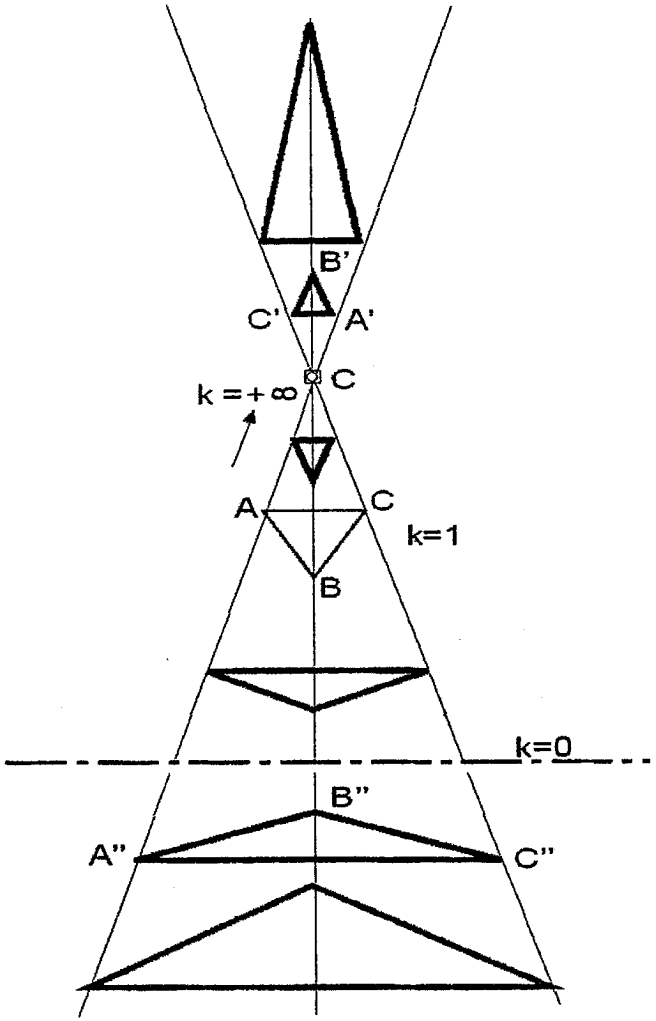


Fig. 47a

Case II

We leave to the reader to watch the behaviour of F in the other two cases (Figs. 47b and 47c).

Case III

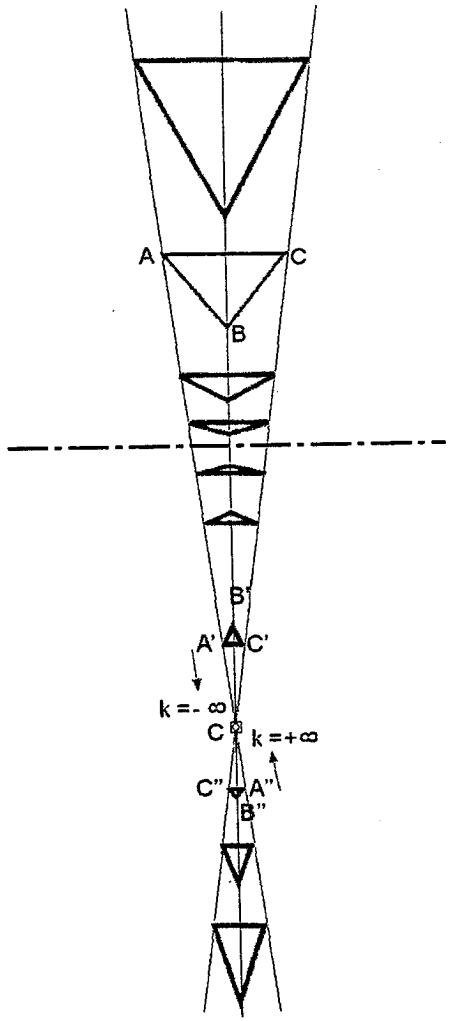


Fig. 47b

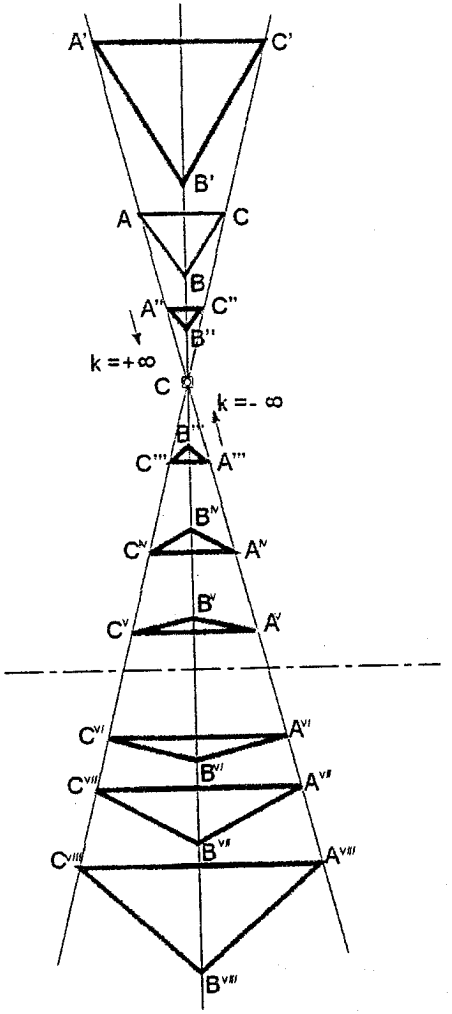


Fig. 47c

A.4.11. Reversion and inversion of the transformed image

Beside the deformations (compressions and dilatations), the image of F transformed through central homology appears, in all the three cases, as "reversed" with respect to the axis and "reversed and inversed" with respect to the center.

A.4.12. Affine transformations

The affine group is a subgroup of the projective group, i.e., the subgroup leaving invariant the straight line from infinity.

An affine transformation in $\mathbf{R}^2$ is given by equations of the following form:

A.4.12a: $x' = ax + by + c \qquad y' = dx + ey + f$

$$\det \begin{matrix} a & b \\ d & e \end{matrix} \neq 0$$

A.4.13. Parallel homology

Parallel homology is the affinity where $b = c = d = f = 0$, $e = 1$, hence this transformation is defined by equations of the form

$$x' = ax, \qquad y' = y$$

Since parallel homology is a homology whose centre is at infinity, the position of the point M', the transform of M, depends on the value of the angle β of the projective line MM_1 with the homology axis A, and on the value of the ratio k

$$k = \frac{M'M_1}{MM_1}^{4}$$

Fig. 48a illustrates the transformation through parallel homology of the figure F in terms of the ratio k value.

Remark. When the projective straight lines are *perpendicular to the axis* and $k = -1$, the transformation generates an *axial symmetry* (Fig. 48b).

[4] The center C being situated at infinity, CM = CM'; as a consequence, the homology biratio

$$\frac{\frac{CM}{MM_1}}{\frac{CM'}{M'M_1}} \text{ becomes } \frac{M'M_1}{MM_1}$$

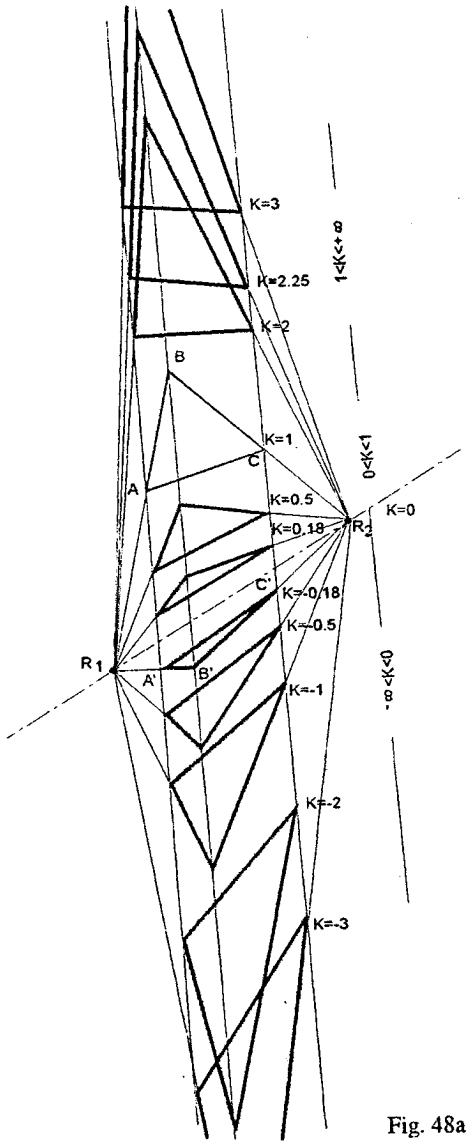


Fig. 48a

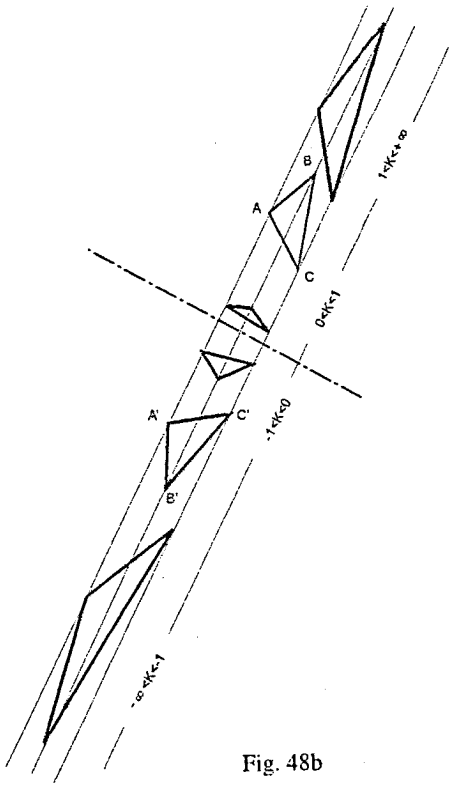


Fig. 48b

A.4.14. Translation

Translation is the affinity where in the equations $a = e = 1$, $b = d = 0$, hence the translation is defined by equations of the following form:

$$x' = x + c \qquad y' = y + f$$

the values $x + c$ and $y + f$ representing the components of a vector referred to as translation vector, completely defining the mapping (Fig. 49).

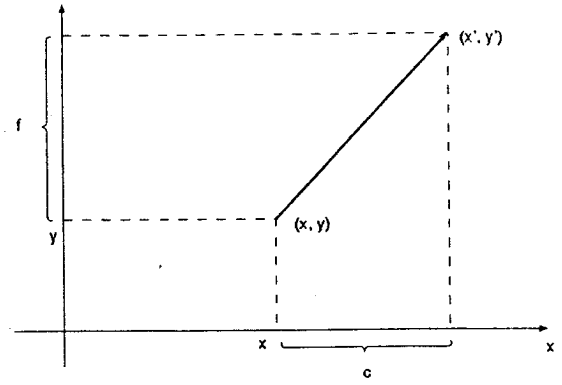


Fig. 49

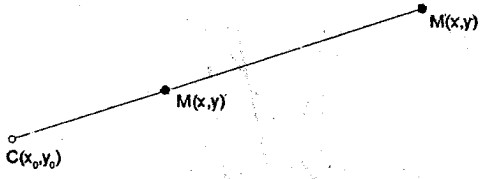
A.4.15. Homothety

Homothety is the affinity having as equations

$$x' = x_0 + k\,(x - x_0), \qquad y' = y_0 + k\,(y - y_0)$$

The point C_0 of coordinates x_0; y_0 is the fixed point in this transformation and is called centre of homothety. The number k is referred to as homothety ratio et represents the CM'/CM ratio of the segments $\overline{CM'}$ and $\overline{CM}$, determined by the centre of homothety C, by a point M in the plane and by its image M' through homothety. The points C, M and M' are collinear (Fig. 50).

Fig. 50



The relative position of the transformed point M' as against the fixed points C and M, is defined by the value of ratio k.

Fig. 51 synthetically shows the position of the transformed point M' relatively to the fixed points C and M depending of the value of k: $(M') = f(k)$

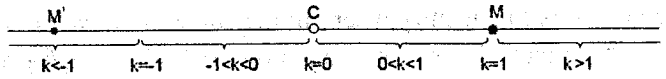


Fig. 51

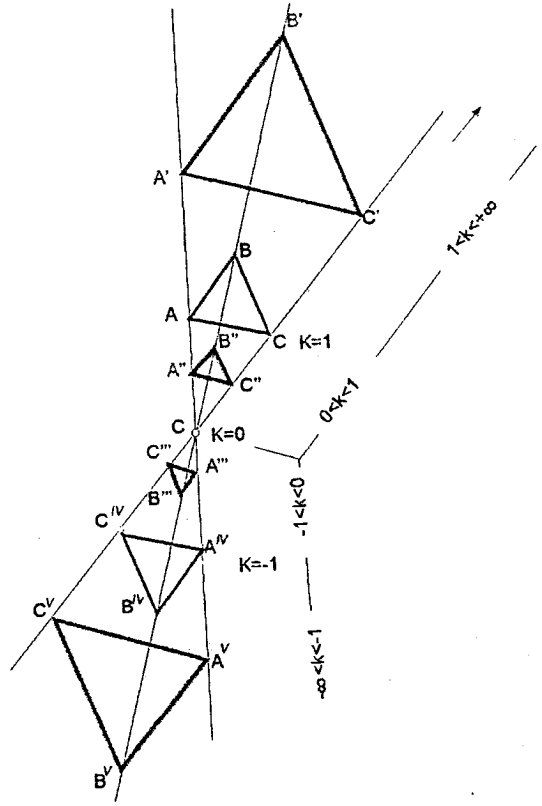
We remark that, for $k = 1$, $M = M'$ (M is transformed into itself), and for $k = 0$, M' is in the centre of homothety. A remarkable property of homothety is the fact that, for $k = -1$, M' becomes symmetric of M relatively to

$$C\ (k = -1,\ \overline{CM} = \overline{CM'})$$

When M is in C $(x = x_0)$, $(y = y_0)$, it is transformed into itself for any value of k: $(x - x_0) = 0 \rightarrow x' = x$, $(y - y_0) = 0 \rightarrow y' = y$.

Fig. 52 illustrates the transformation through homology of F depending on the value of the homothety ratio k.

Fig. 52



For $k = 1$, F transforms into itself. For $1 < k < \infty$, F *dilates*. For $1 > k > 0$, F is compresses (as $k \rightarrow 0$). For $k = 0$, all the points of F aggregate in the homothety centre. For $0 > k > -1$, the figure is *compressed, reverted* and *inverted*. For $-1 > k > -\infty$, F is *reverted, inverted* and *expanded* (as $k \rightarrow -\infty$).

Remark. For $k = -1$, the homothety generates a *central symmetry*.

A.4.16. Euclidean transformations

The Euclidean transformations are defined by the equations

$$X' = ax' + by + c$$
$$Y' = dx' + cy = f$$

with $a^2 + d^2 = b^2 + e^2 \neq 0$

Rotation

One of the basic transformations of the Euclidean group is the *rotation* in $\mathbb{R}^2$, defined by the following equations

$$x' - x_0 = \cos\alpha\,(x - x_0) - \sin\alpha\,(y - y_0)$$
$$y' - y_0 = \sin\alpha\,(y - y_0) + \cos\alpha\,(x - x_0)$$

The point C of coordinates x_0, y_0 is fixed in this transformation; it bears the name of centre of rotation. The parameter α is referred to as rotation angle. By this transformation the plane points describe arcs of circle having the centre in C, and the angle at the center of magnitude α.

A vector in the plane, subjected to rotation, changes its direction and sense depending on the rotation angle α (Fig. 53).

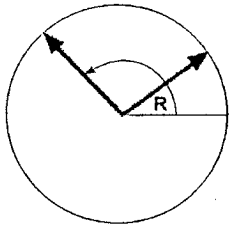


Fig. 53

A.4.17. Inversion

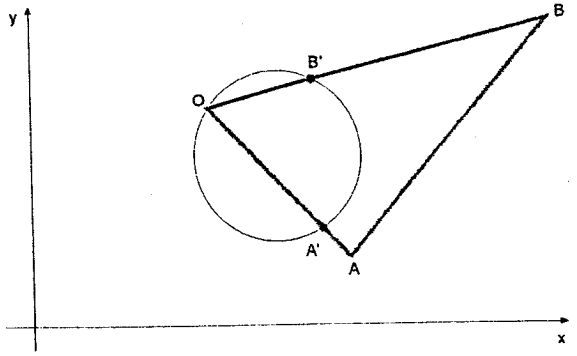
With respect to a rectangular axes system having the origin in the pole of inversion, between the coordinates of a point A (x, y) and its transformed through inversion A' (x', y'), the following relationships exist:

$$x' = \frac{k^2 x}{x^2 - y^2} \qquad y' = \frac{k^2 y}{x^2 - y^2}$$ (see A.3.13).

Inversion of a segment AB relative to the inversion circle crossing the pole O

The segment AB is exterior to the inversion circle. The transformed segment $A'B'$ is on an interior inversion circle and traverses O, and its radius is $\varphi = R$ (Fig. 54).

Fig. 54



The segment AB is tangent to the inversion circle. The transformed segment is on a circle crossing O whose radius is $\varphi = R/2$ (Fig. 55).

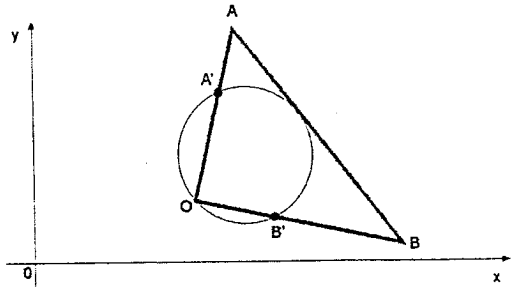


Fig. 55

The segment AB is secant to the inversion circle. The transformed segment is on a circle whose radius $R/2 < \varphi < \infty$ crosses the point O. The transforms of the segment fragment AB inside the circle O (inversion circle) is outside of it, and that outside the circle O is inside it.

The points of the segment AB belonging to the inversion circle O also, are transformed into themselves (Fig. 56).

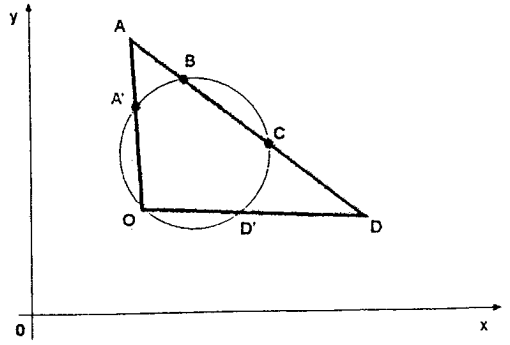


Fig. 56

The segment AB crosses the point O. The transformed segment $A'B'$ is on the straight line AB (Fig. 57).

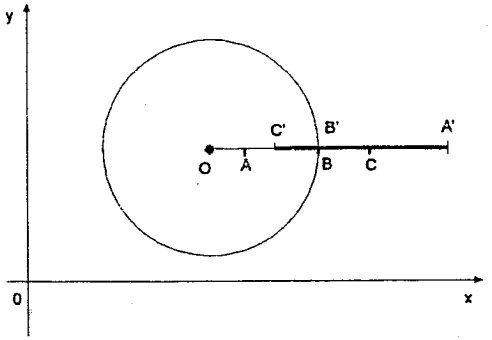


Fig. 57

For any estimate of R, the point O is sent to infinity.

The studied examples analyse the inversion with sign +. For an inversion with the sign –, the transformed segment $A'B'$ is symmetrical with the transformed segment $A'B'$ through inversion with the sign + of the segment AB, related to the centre O of the inversion circle.

Inversion is a particular case of topological transformation preserving neighbourhoods and transforming lines into circles or lines and circles into lines or circles.

A.4.18. Metric transformations

The metric transformations are defined by the equations

$$x' = ax + by + c$$
$$y' = dx + ey + f$$

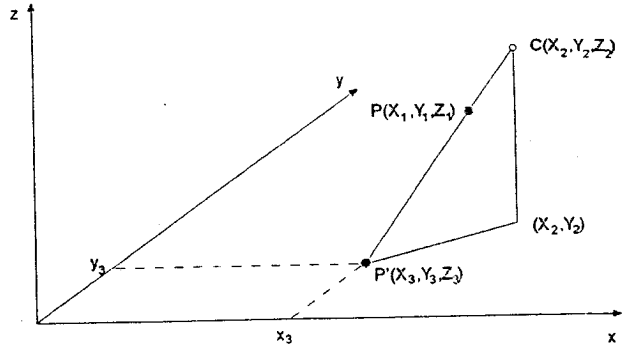
with $a^2 + d^2 = b^2 + e^2 = 1$.

A.4.19. Projection from $\mathbf{R}^3$ in $\mathbf{R}^2$

A point P of coordinates x_1, y_1, z_1 and a centre of projection C of coordinates x_2, y_2, z_2, being given in $\mathbf{R}^3$, the projection of P from C in $\mathbf{R}^2$ is represented by the point $P' \in \mathbf{R}^2$ of coordinates x_3, y_3 (Fig. 58).

$$x_3 = x_2 - \frac{z_2 (x_2 - x_1)}{z_2 - z_1} \quad ; \qquad y_3 = y_2 - \frac{z_2 (y_2 - y_1)}{z_2 - z_1}$$

Fig. 58



A.4.20. Behaviour of a geometrical plane figure situated in the space $\mathbf{R}^3$ under the action of transformation through projection from $\mathbf{R}^3$ in $\mathbf{R}^2$

Speaking of transformation through projection of a coplanar set of points – especially of a plane figure F– from $\mathbf{R}^3$ in $\mathbf{R}^2$, depending on the position of the projection centre Cp as related to F, the following cases are remarked.

(a) $z\ Cp \to +\infty$. (Cp situated at $+\infty$ (cylindrical projection)).

When F is situated in a plane parallel to $\mathbf{R}^2$ and when the projecting lines are orthogonal (Cp is situated at infinity in the Oy axis direction), F', the image through projection of F in $\mathbf{R}^2$ is an identical transformation: F' is congruent with F (Figs. 59a and 59b).

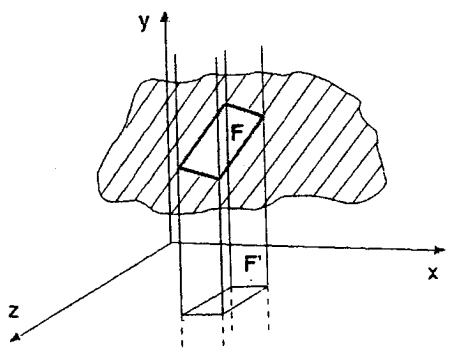


Fig. 59a

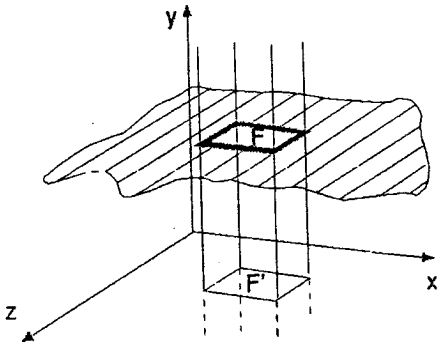


Fig. 59b

(b) $+\infty > z\ Cp > z\ F$ (the center locates "above" F at finite distance).

F' appears as "expanded" relatively to the image of F obtained through cylindrical orthogonal projection (Fig. 60).

Both the congruence in the preceding case and the expansion in the present case are intuitively discernible and easily proved geometrically (Thales theorem), when F consists of a set of linearly arranged points (as line segments).

Generally, the dilatation through conical projection may be expressed by the following relations:

Two points $M \in F$ and $M' \in F'$ being given

$$x\,Cp = xM \Rightarrow xM' = xM$$
$$x\,Cp < xM \Rightarrow xM' > xM$$
$$x\,Cp > xM \Rightarrow xM' > xM$$
$$y\,Cp = yM \Rightarrow yM' = yM$$
$$y\,Cp < yM \Rightarrow yM' > yM$$
$$y\,Cp > yM \Rightarrow yM' < yM$$

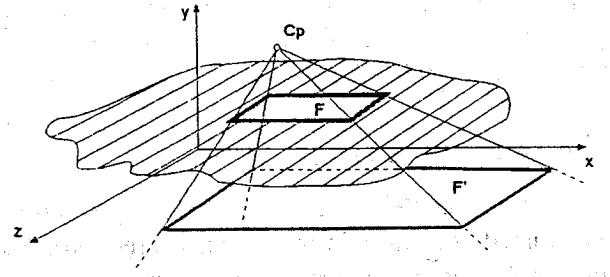


Fig. 60

(c) $zF > zCp > 0$ (Cp lies between F and $\mathbf{R}^2$) (Fig. 61a, b, c).

F' is reversed and inverted as against the parallel and orthogonal projection.

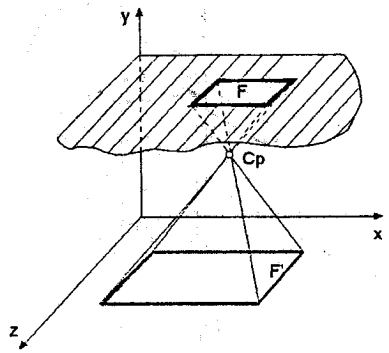


Fig. 61a

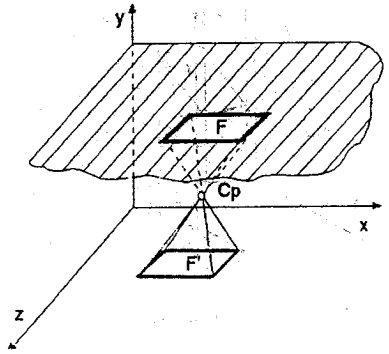
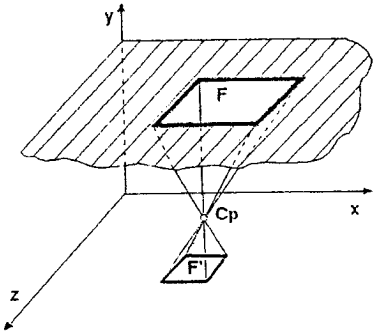


Fig. 61b

Fig. 61c



If $zCp > \dfrac{zF}{2}$, F' is expanded (Fig. 61a); if $zCp = \dfrac{zF}{2}$ the image is a *central symmetry* (Fig. 61b); if $zCP < \dfrac{zF}{2}$ the image is compressed (in the table under item (b), the inequality signs are inverted) (Fig. 61c). F' is compressed as $zCp \to 0$.

(d) $zCp = 0$. The projection centre lies in the space $\mathbf{R}^2$ (Fig. 62). F' starts to "expand again" as

$$zCp \to -\infty.$$

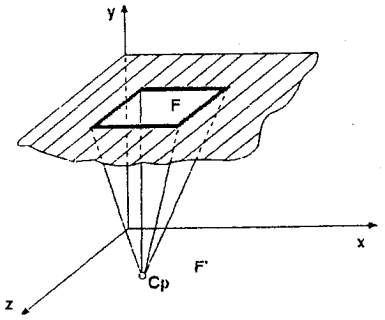


Fig. 62

(e) $0 > zCp > -\infty$.

Cp is situated "under" the space $\mathbf{R}^2$ (Fig. 63).

F' begin to "redilate" when $zCp \to -\infty$.

(f) $zCp \to \infty$. Cp is situated at $-\infty$ (cylindrical projection). When Cp is directed towards F in the Oy direction, the projection lines are parallel and orthogonal and when F is in a parallel plane to $\mathbf{R}^2$, F' is congruent with F (Fig. 64).

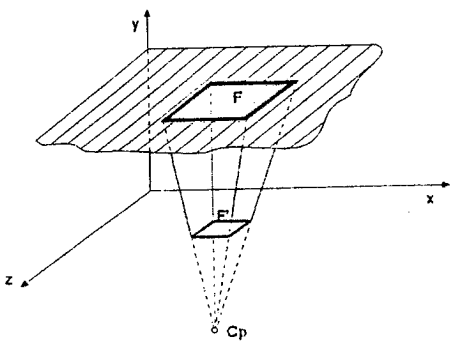


Fig. 63

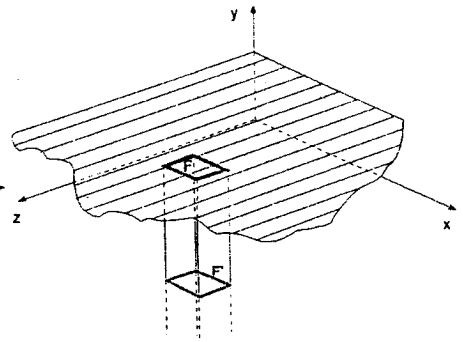


Fig. 64

A.4.21. Rotation in R^3

In the following we will consider a figure in a plane L parallel to the plane xOy. Let us study the rotation of a point belonging to the figure around an axis A included in L. The figure rotation comes down to the rotation of the points defining it: for a line segment the segment ends, for a triangle its vertices, for a rectangle its corners, etc. This method holds for any axis also.

Given

z_0 height of plane L

M a point of the figure $M = xM$
yM
zM

A a point crossed by the rotation axis $A = xA$
yA
zA

α angle of the rotation axis with Ox

β rotation angle of point M around the axis A.

Required M' point obtained through rotation M around A at an angle β (Fig. 65).

The point B is determined on the rotation axis at the distance dx from point A

$$B = xB\ yB\ zB$$

If $\beta \neq 90°$, $xB = xA + dx$, $yB = yA + tg\ \beta \cdot dx$

If $\beta = 90°$, $xB = xA$, $yB = yA + dx$.

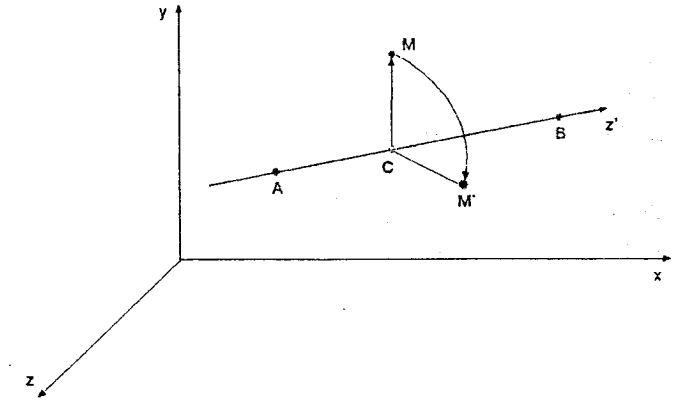


Fig. 65

The point M rotates around the axis according to the right-hand rule looking from A to B.[5]

We recall that when two vectors: $x = x_1$ and $y = y_1$
x_2 y_2
x_3 y_3

are given, their sum is defined by $x + y = x_1 - y_1$
$x_2 - y_2$
$x_3 - y_3$

and their scalar product by

$$x * y = x_1 * y_1 + x_2 * y_2 + x_3 * y_3$$

The norm of a vector is defined by

$$x = x_1^2 + x_2^2 + x_3^2$$

As a method, we will first rotate M in a plane π which passes through M and is perpendicular to the axis, then we will carry out a transformation of coordinates.

The axis meets the plane π in the point C

$$p = \frac{(A-B)\cdot(M-A)}{|A-B|^2}$$

[5] By rotation according to the right-hand rule we mean the rotation sense of the right-hand fingers perpendicular to the axis in motion from the "open" (palm) position, towards the "closed" (fist) position, when the thumb is directed from A towards B.

$C = A + p\,(A - B)$	coordinates of M' with
$r = \lvert M - C \rvert$	respect to the mark
$(xp = r \cos\alpha)$	determined by the
(r $\sin\alpha$)	axis AB and the
(O)	plane π

In order to compute the coordinates of M' relatively to the $Oxyz$ mark, we will determine a transformation matrix as follows:

$$V = \begin{matrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{matrix}$$

$$\begin{matrix} v_{11} \\ v_{21} = M - C \\ v_{31} \end{matrix} \qquad \begin{matrix} v_{13} \\ v_{23} = B - A \\ v_{33} \end{matrix}$$

$$v_{12} = \frac{v_{21}v_{23}}{v_{23}v_{33}} \qquad v_{22} = \frac{v_{11}v_{13}}{v_{23}v_{33}} \qquad v_{23} = -\frac{v_{11}v_{13}}{v_{21}v_{23}}$$

Still V cannot be used as transformation matrix, since it is not yet normalized.

The matrix T is determined as follows:

$$T_{ij} = \frac{v_{ij}}{v_{1j}^2 + v_{2j}^2 + v_{3j}^2}$$

$1 \le i \le 3$

$1 \le j \le 3$

Thus $M' = T*Xp + C$

Remark. When any axis is desired, it suffices to indicate the respective axis by two points, A and B, the axis being considered as oriented from A to B.